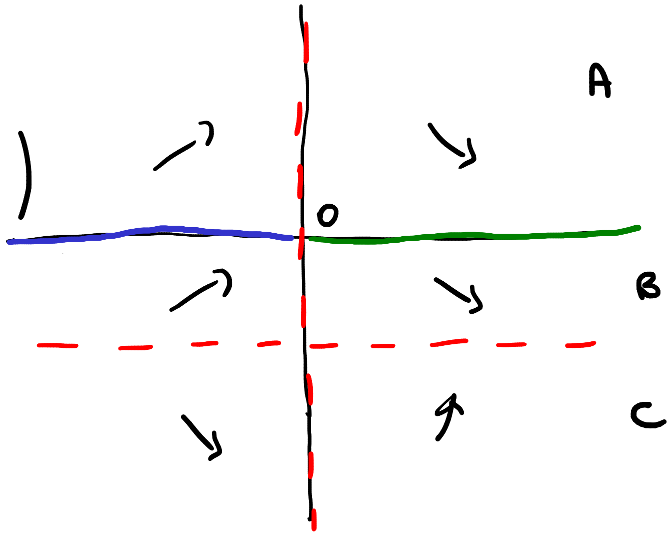
5 dicembre 2023

Riprendo

$$y' = -\frac{y^2}{t(y+1)} = \frac{1}{t}\left(-\frac{y^2}{y+1}\right)$$



TEUL ... ✓

$\Omega : \mathbb{R}^* \times \mathbb{R} \setminus \{-1\}$

Cerco sol. non costanti:

$$H'(y) = -\frac{y+1}{y^2} = -\frac{1}{y} - \frac{1}{y^2} \qquad G'(t) = \frac{1}{t}$$

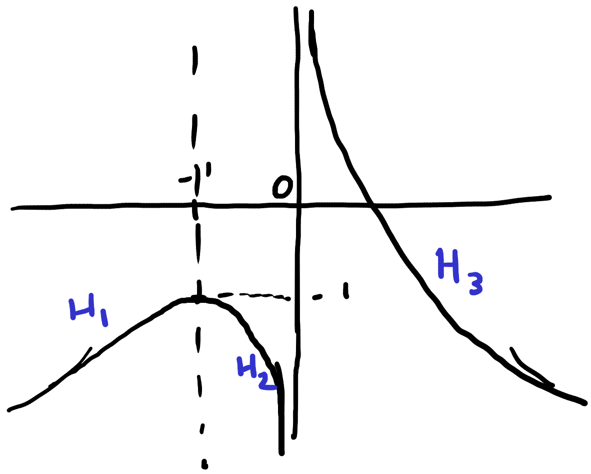
$$H(y) = -\ln|y| + \frac{1}{y} \qquad G(t) = \ln|t|$$

$y = y(t)$ sol. dell' eq :

$$\frac{1}{y(t)} \underbrace{- \ln|y(t)| = \ln|t|}_{???} + c \qquad c \in \mathbb{R}$$

"Studio" H :

$$H(y) = \frac{-y\ln|y| + 1}{y}$$



$$\lim_{y \to 0^\pm} H(y) = \pm\infty$$

H'

	-1		0	
$+$		$-$		$-$

$H(-1) = -1$

	dom.	imm.	monot
H_1	$]-\infty,-1[$	$]-\infty,-1[$	↑

$$\varphi_c(t) := H_1^{-1}(\ln|t| + c)$$

$H_2 \quad]-1,0[\quad]-\infty,-1[\quad \downarrow \qquad \psi_c(t) = H_2^{-1}(\ln|t|+c)$

$H_3 \quad]0,+\infty[\quad \mathbb{R} \quad \downarrow \qquad \eta_c(t) = H_3^{-1}(\ln|t|+c)$

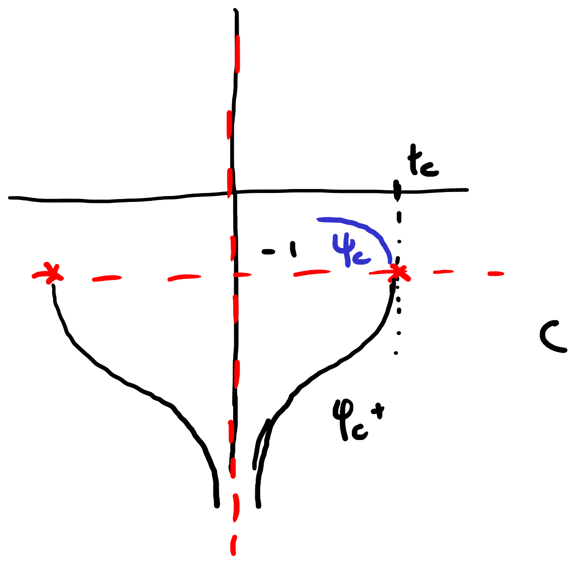
φ_c : ha valori in $]-\infty,-1[$ ("vive in C")

Condizione di esistenza:

$$\ln|t| + c < -1 \iff \ln|t| < -(c+1)$$
$$\iff 0<|t| < e^{-(c+1)} =: t_c$$

$\Rightarrow$ due soluzioni:

$$\varphi_c^+(t) = \varphi_c(t) \qquad t \in]0, t_c[$$
$$\varphi_c^-(t) = \varphi_c(t) \qquad t \in]-t_c, 0[$$



$$\varphi_c(t) = H_1^{-1}(\ln|t| + c)$$

$$\lim_{t\to 0^+} \varphi_c^+(t) = \lim_{s\to -\infty} H_1^{-1}(s) = -\infty$$

$$\lim_{t\to t_c^-} \varphi_c^+(t) = \lim_{s\to -1} H_1^{-1}(s) = -1$$

$$y' = -\frac{y^2}{t(y+1)}$$

Oss: $(\varphi_c^+)'(t) \to +\infty$ per $t \to t_c^-$

$$\psi_c(t) = H_2^{-1}(\ln|t| + c)$$

Condizione di esistenza: come per φ_c

$\Rightarrow$ due soluzioni $(\forall c \in \mathbb{R})$

$$\psi_c^+(t) = \psi_c(t) \qquad t \in]0, t_c[$$
$$\psi_c^-(t) = \psi_c(t) \qquad t \in]-t_c, 0[$$
"vivono in B"

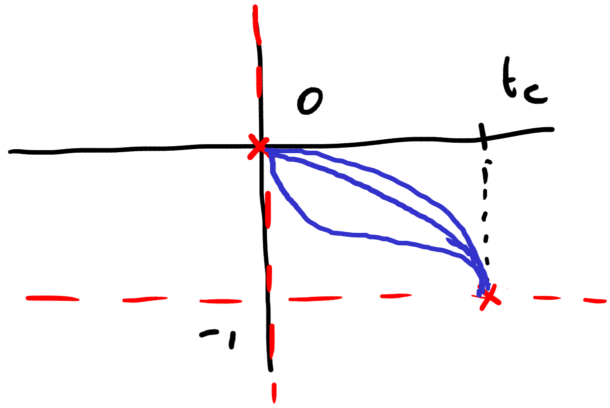
$$\lim_{t \to t_c^-} \psi_c^+(t) = \lim_{s \to -1} H_2^{-1}(s) = -1$$

$$(\psi_c^+)'(t) \to -\infty$$

$$\lim_{t \to 0^+} \psi_c^+(t) = \lim_{s \to -\infty} H_2^{-1}(s) = 0$$

$$y' = -\frac{y^2 \to 0}{t\,(y+1)} \qquad t \to 0^+,\ (y+1) \to 1$$

Forma di indecisione!



Parentesi: cerco di capire con quale pendenza la soluzione si avvicina a 0

Suppongo che $\exists \lim_{t \to 0^+} \psi_c'(t) =: \alpha \in \mathbb{R}$

$$\alpha = \lim_{t \to 0^+} \psi_c'(t) = \lim_{t \to 0^+} -\frac{(\psi_c(t))^2}{t\,(\underbrace{\psi_c(t)}_{\to 0}+1)} \quad \to 1$$

$$= \lim_{t \to 0^+} -\frac{(\psi_c(t))^2}{t}$$

$$\overset{H}{=} \lim_{t \to 0^+} -\frac{2\,\overbrace{\psi_c(t)}^{\to 0}\;\overbrace{\psi_c'(t)}^{\to \alpha \in \mathbb{R}}}{1} = 0$$

$$\Rightarrow \alpha = 0.$$

Per decidere tra $\alpha = 0$ e $-\infty$ esamino la convessità/concavità della soluzione:

$$\forall t: \quad y'(t) = - \frac{y(t)^2}{t(y(t)+1)}$$

$$\Rightarrow \quad y''(t) = - \frac{2y(t)\overbrace{y'(t)\, t(y(t)+1)} - y(t)^2\left(y(t)+1+t\,y'(t)\right)}{\left(t(y(t)+1)\right)^2}$$

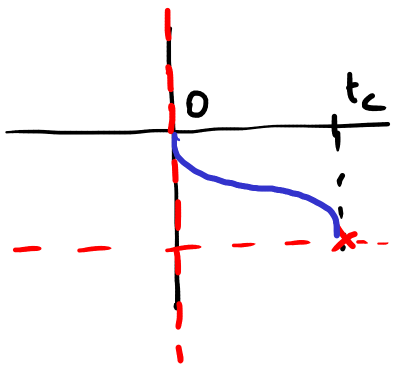
$$= - \frac{2y(t)\left(-y(t)^2\right) - y(t)^2\left(y(t)+1-\frac{y(t)^2}{y(t)+1}\right)}{(\ldots)^2}$$

$$= \underbrace{\frac{y(t)^2}{(\ldots)^2}}_{>0}\left[\underbrace{2y(t)}_{\to 0} + \underbrace{y(t)}_{\to 0} + \underbrace{1}_{>0} - \underbrace{\frac{y(t)^2}{y(t)+1}}_{\to 0}\right]$$

$\Rightarrow$ per t vicino a 0^+:
$y''(t) > 0$.

$\Rightarrow$ ψ_c^+ in un intorno destro di $t=0$ è convessa

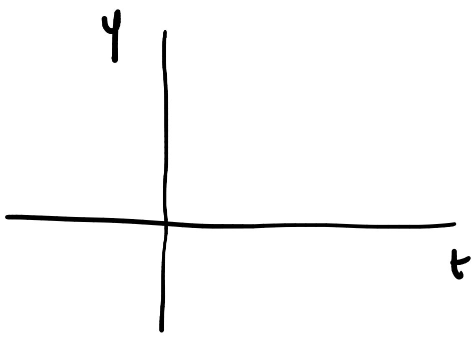
$$\Rightarrow \quad (\psi_c^+)'(t) \underset{t \to 0^+}{\longrightarrow} -\infty$$



In alternativa:

$$\frac{1}{y(t)} - \ln|y(t)| = \ln|t| + c$$

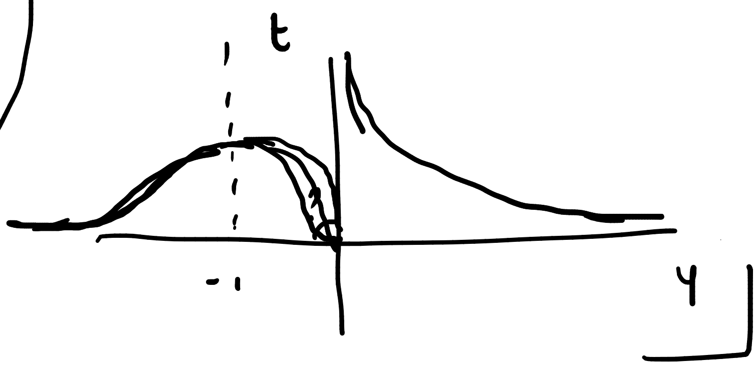
$$\frac{1}{y} - \ln|y| = \ln|t| + \underbrace{c}_{\in \mathbb{R}}$$



$$\ln|t| = \frac{1}{y} - \ln|y| + \underbrace{c'}_{\in \mathbb{R}}$$

$$|t| = e^{\frac{1}{y} - \ln|y| + c'} = \frac{c\, e^{\frac{1}{y}}}{|y|}$$

$$t = \pm \frac{c\, e^{\frac{1}{y}}}{|y|}$$



Concludo:

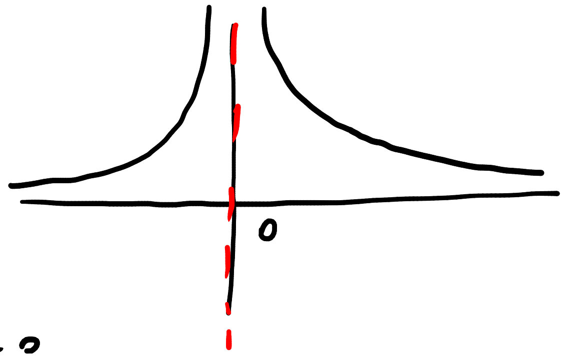
ha valori in $]0,+\infty[$

$$\eta_c(t) = H_3^{-1}(\ln|t| + c)$$

definita in $\mathbb{R}$

$\Rightarrow$ due sol: $\eta_c^{\mp}$ definite in $]-\infty,0[$, $]0,+\infty[$

$$\lim_{t\to 0^+} \eta_c^+(t) = \lim_{s\to -\infty} H_3^{-1}(s) = +\infty$$



$$\lim_{t\to +\infty} \eta_c^+(t) = \lim_{s\to +\infty} H_3^{-1}(s) = 0$$

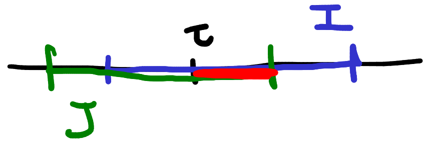
Dimostro il teor. sul prolungamento massimale

1. Siano $\varphi : I \to \mathbb{R}^n$ e $\psi : J \to \mathbb{R}^n$ soluzioni di

$$(*) \quad y' = f(t,y)$$

Suppongo $I \cap J \neq \emptyset$ e che esista $\tau \in I \cap J$

t.c. $\varphi(\tau) = \psi(\tau)$.



Voglio dimostrare che $\varphi(t) = \psi(t)$ $\forall t \in I \cap J$.

Suppongo $\tau < \sup(I \cap J)$ e dimostro che

$$\varphi(t) = \psi(t) \quad \forall t \in I \cap J \cap]\tau, +\infty[$$

Definisco $E = \{ t \in I \cap J \cap]\tau, +\infty[\mid \varphi(t) \neq \psi(t) \}$

Voglio dimostrare che $E = \emptyset$.

Per assurdo, suppongo $E \neq \emptyset$

Osservo che $E \subset]\tau, +\infty[$, quindi è limitato inferiormente.

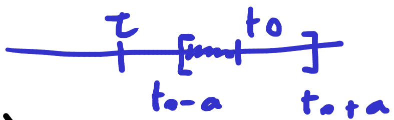
Pertanto, posto $t_0 := \inf E$: $t_0 \in \mathbb{R}$.

In effetti: $t_0 \geq \tau$.

Verifico che $\varphi(t_0) = \psi(t_0)$. $\odot$

Se $t_0 = \tau$: $\odot$ è vera per ipotesi.

Se $t_0 > \tau$:



suppongo $\varphi(t_0) \neq \psi(t_0)$; dato che φ e ψ sono continue, per teor. della permanenza del segno esiste $a > 0$ t.c.

$$\varphi(t) \neq \psi(t) \quad \forall t \in [\underbrace{t_0 - a}_{> \tau}, t_0 + a]$$

Allora: $[t_0 - a, t_0] \subseteq E$

$$\Rightarrow t_0 = \inf E \leq \inf [t_0 - a, t_0] = t_0 - a$$

$$t_0 \leq t_0 - a, \quad a > 0 \quad !!!$$

Dunque: $\varphi(t_0) = \psi(t_0)$ in entrambi i casi.

Pongo $x_0 := \varphi(t_0) = \psi(t_0)$ e considero il PdC

$$(**)\begin{cases} y' = f(t,y) \\ y(t_0) = x_0 \end{cases}$$

Osservo che φ e ψ risolvono $(**)$

Per il TEUL:

$$\exists\, \delta > 0 \text{ t.c. } \varphi(t) = \psi(t) \quad \forall t \in [t_0 - \delta, t_0 + \delta].$$

Però: $t_0 = \inf E$, $t_0 + \delta > t_0$

$$\Rightarrow \exists\, \hat{t} \in E \text{ t.c. } \underbrace{t_0 \le \hat{t} < t_0 + \delta}$$

$$\Downarrow$$

!!! $\varphi(\hat{t}) = \psi(\hat{t})$

$\square$

(2) Sia (φ, I) sol. di $y' = f(t,y)$ $(*)$

Definisco

$$\mathcal{P} = \{ (\psi, J) \mid (\psi, J) \text{ risolve } (*),\ I \subseteq J,\ \psi_{|I} = \varphi \}$$

Nota: $(\varphi, I) \in \mathcal{P} \Rightarrow \mathcal{P} \neq \emptyset$

Definisco

$$a := \inf \{ \inf J \mid (\psi, J) \in \mathcal{P} \}$$

$$b := \sup \{ \sup J \mid (\psi, J) \in \mathcal{P} \}$$

Oss: $a \le \inf J < \sup J \le b$

$\Rightarrow \ ]a,b[\ \neq \emptyset$

Definisco $\eta :]a,b[\to \mathbb{R}^n$ in questo modo:

$\forall t \in]a,b[\ : \ \eta(t) := \psi(t)$

dove $(\psi, J) \in P$ e $t \in J$

Noto che η è ben posta: fissato $\bar{t} \in]a,b[$

se (ψ, J) e $(\tilde{\psi}, \tilde{J})$ sono tali che

$\bar{t} \in J$ e $\bar{t} \in \tilde{J}$, necessariamente

$$\psi(\bar{t}) = \tilde{\psi}(\bar{t})$$

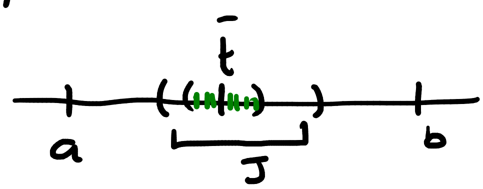
perché ψ e $\tilde{\psi}$ coincidono con φ e dunque tra loro in I, e quindi per ① coincidono in tutta l'intersezione $J \cap \tilde{J}$, e in particolare in $\bar{t}$.

Per costruzione: $\forall t \in]a,b[\ : (t, \eta(t)) \in \Omega$

Devo mostrare che η è derivabile e soddisfa l'equazione.

Fisso $\bar{t} \in]a,b[$; scelgo (ψ, J) t.c. $\bar{t} \in \mathrm{int}(J)$

Esiste un intorno U di $\bar{t}$ t.c. $\forall t \in U$: posso usare ψ per definire η, cioè:



$$\eta(t) = \psi(t) \quad \forall t \in U$$

$$\Rightarrow \quad \forall t \in U: \quad \eta'(t) = \psi'(t)$$

$$\Rightarrow \quad \underline{\eta'(\bar{t})} = \psi'(\bar{t}) = f(\bar{t}, \psi(\bar{t})) = \underline{f(\bar{t}, \eta(\bar{t}))}$$

Dunque: η risolve $\circledast$

Ovviamente $(\eta,]a,b[)$ è prolungamento di (φ, I)

$\forall t \in I$: $\quad \eta(t) = \psi(t) = \varphi(t)$

per una certa $(\psi, J) \in \mathcal{P}$

Per definizione di a e b, η non ha prolungamenti e quindi è massimale.

L'unicità del prol. mass. deriva da ①

$\square$

Intermezzo: $\quad f(x) = (1+x)^\alpha \quad x \in]-1, +\infty[$

$\alpha \in \mathbb{R} \setminus \mathbb{N}$.

Oss: $f \in C^\infty$

$\forall x \in]-1, +\infty[$:

$$f'(x) = \alpha(1+x)^{\alpha-1}$$
$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$
$$\vdots$$
$$f^{(n)}(x) = \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n}$$

Coeff. di Taylor di centro 0: $=: \binom{\alpha}{n}$

$$\underset{\binom{\alpha}{0}}{1}, \quad \underset{\binom{\alpha}{1}}{\alpha}, \quad \underset{\binom{\alpha}{2}}{\frac{\alpha(\alpha-1)}{2!}}, \quad \dots, \quad \boxed{\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}}, \dots$$

$\Rightarrow$ Serie di Taylor:

$$\sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n \qquad \binom{\alpha}{n} \neq 0 \text{ perché } \alpha \notin \mathbb{N}$$

Determino il r.d.c. con il criterio di D'Alembert:

$$\frac{\left|\binom{\alpha}{n+1}\right|}{\left|\binom{\alpha}{n}\right|} = \frac{|\cancel{\alpha(\alpha-1)}\dots\cancel{(\alpha-n+1)}(\alpha-n)|}{\underset{n+1}{\cancel{(n+1)!}}} \cdot \frac{\cancel{n!}}{|\cancel{\alpha(\alpha-1)}\dots\cancel{(\alpha-n+1)}|}$$

$$= \frac{|\alpha-n|}{n+1} \xrightarrow[n\to+\infty]{} 1 \quad \Rightarrow \quad R=1$$

Definisco la somma della serie di Taylor ponendo

$$T(x) := \sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n \qquad \forall x \in]-1,1[.$$

Voglio dimostrare che $T(x) = (1+x)^\alpha \quad \forall x \in]-1,1[$.

Derivo T termine a termine:

$\forall x \in]-1,1[$: $\quad T'(x) = \sum_{n=1}^{+\infty} \binom{\alpha}{n} n x^{n-1}$

Calcolo:

$$(1+x)T'(x) = \sum_{n=1}^{+\infty} \binom{\alpha}{n} n x^{n-1} + \sum_{n=1}^{+\infty} \binom{\alpha}{n} n x^n$$

$$= \sum_{n=0}^{+\infty} \binom{\alpha}{n+1}(n+1)\, x^n + \sum_{n=1}^{+\infty} \binom{\alpha}{n} n\, x^n$$

$$= \alpha + \sum_{n=1}^{+\infty} \Big[\underbrace{\binom{\alpha}{n+1}(n+1) + \binom{\alpha}{n} n}_{\odot} \Big] x^n =$$

$$\odot = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)}{\cancel{(n+1)!}\ n!} \cancel{(n+1)} + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} n$$

$$= \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \left(\alpha - n + n \right) = \binom{\alpha}{n} \alpha$$

$$= \alpha + \sum_{n=1}^{+\infty} \binom{\alpha}{n} \alpha\, x^n = \alpha \Big[\underbrace{1}_{\binom{\alpha}{0}} + \sum_{n=1}^{+\infty} \binom{\alpha}{n} x^n \Big]$$

$$= \alpha \sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n = \alpha\, T(x)$$

Quindi: $\forall x \in]-1, 1[: \quad (1+x) T'(x) = \alpha\, T(x)$

Cioè: T risolve l'eq. diff.

$$(1+x)\, y' = \alpha\, y$$

Nota: qui la var. indip. si chiama x

$$y' = \frac{\alpha}{1+x}\, y$$

$$H'(y) = \frac{1}{y} \Rightarrow H(y) = \ln |y|$$

$$G'(x) = \frac{\alpha}{1+x} \Rightarrow G(x) = \alpha \ln(1+x) = \ln (1+x)^\alpha$$

$$\Rightarrow \ln |y(x)| = \ln (1+x)^\alpha + c', \quad c' \in \mathbb{R}$$

$$\Rightarrow \quad |y(x)| = c\,(1+x)^{\alpha} \qquad c>0$$

Osservo che $T(0) = \underbrace{1}_{>0}$; sostituendo:

$$|T(0)| = c\,(1+0)^{\alpha}$$

$$\Rightarrow \quad c = 1$$

$$\Rightarrow \quad |\underbrace{T(x)}_{>0\ \forall x}| = (1+x)^{\alpha}$$

$$\Rightarrow \quad T(x) = (1+x)^{\alpha} \qquad \forall x \in \left]-1,1\right[$$

$\square$

Equazioni di Manfredi

$$y' = \frac{t^3+y^3}{t y^2}$$

$$= \frac{t^2}{y^2} + \frac{y}{t}$$

$$= \frac{1}{\left(\frac{y}{t}\right)^2} + \frac{y}{t}$$

$$g(s) = \frac{1}{s^2} + s$$

$$\to \quad y' = g\left(\frac{y}{t}\right)$$

$$y' = \frac{y}{t}\ln\left(\frac{y}{t}\right)$$

$$g(s) = s\ln(s)$$

$$y' = g\left(\frac{y}{t}\right)$$

$$y' = g\left(\frac{y}{t}\right)$$

$$z(t) := \frac{y(t)}{t} \quad \Rightarrow \quad y(t) = t\,z(t)$$

$$z + t z' = g(z)$$

$$t z' = g(z) - z$$

$$\left| \; z' = \frac{g(z) - z}{t} \; \right|$$

$t \neq 0$

$$y' = \frac{t^3 + y^3}{t y^2} = \frac{1}{\left(\frac{y}{t}\right)^2} + \frac{y}{t}$$

$$z = \frac{y}{t}$$

$$y = t z$$

$$z + t z' = \frac{1}{z^2} + z$$

$$t z' = \frac{1}{z^2} \qquad z' = \frac{1}{t} \cdot \frac{1}{z^2}$$

$$H'(z) = z^2 \implies H(z) = \frac{z^3}{3}$$

$$G'(t) = \frac{1}{t} \implies G(t) = \ln|t|$$

$$\frac{z^3}{3} = \ln|t| + c$$

$$z^3 = 3(\ln|t| + c)$$

$$\frac{y^3(t)}{t^3} = 3(\ln|t| + c) \qquad c \in \mathbb{R}$$

$$y^3(t) = 3t^3(\ln|t| + c)$$

$$y_c(t) = \sqrt[3]{3t^3(\ln|t| + c)} = t\sqrt[3]{3(\ln|t| + c)}$$

DA COMPLETARE