28 novembre 2023

3° quesito esonero:

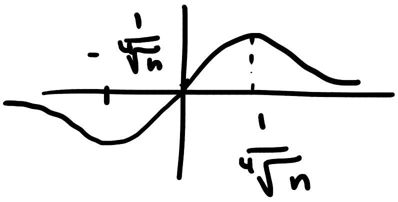
$$f_n(x) = \frac{x}{3+nx^4} \qquad x \in [-1,1]$$

$$(C^0([-1,1],\mathbb{R}), \|\cdot\|_{C^0}) \quad e \quad (C^1([-1,1],\mathbb{R}), \|\cdot\|_{C^1}) \quad ②$$

$$\exists f \in C^0([-1,1],\mathbb{R}) \text{ t.c. } \|f_n - f\|_{C^0} \to 0 \;? \quad ①$$

$$\forall x: \qquad f_n(x) \to 0 =: f(x)$$

① : $\|f_n - 0\|_{C^0} \to 0$? $\iff$ $f_n \to 0$ unif. in $[-1,1]$?



$$\sup_{[-1,1]} |f_n - \overset{\overset{0}{\shortparallel}}{f}| = f_n\left(\frac{1}{\sqrt[4]{n}}\right) = \frac{1}{4\sqrt[4]{n}} \to 0$$

sì! $f_n \to f \equiv 0$ in C^0

② $\iff$ $\|f_n - \overset{\overset{0}{\shortparallel}}{f}\|_{C^1} \to 0$?

$$\underbrace{\|f_n - f\|_{C^0}}_{\substack{\to 0 \\ \text{già visto}}} + \underbrace{\|f_n' - f'\|_{C^0}}_{?} \overset{?}{\to} 0$$

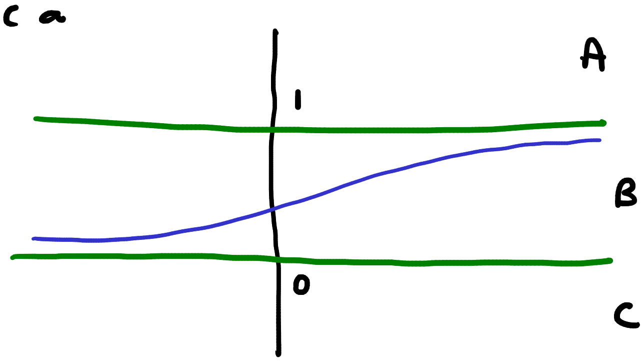
La domanda è: $f_n' \to 0$ unif. in $[-1,1]$?

$$f_n'(x) = \frac{3+nx^4 - x \cdot 4nx^3}{(3+nx^4)^2} = \frac{3(1-nx^4)}{(3+nx^4)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} \frac{1}{3} & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \square \quad \left(\begin{array}{l}\text{La risposta} \\ \text{è: No!}\end{array}\right)$$

Riprendo l'eq. logistica

$y' = y(1-y)$



Sol. in A o C:

...

$$(*) \quad \left| \frac{y(t)}{1-y(t)} \right| = c e^t \qquad c > 0$$

$$\Leftrightarrow \quad \frac{y(t)}{y(t)-1} = c e^t$$

$$y(t) = c e^t y(t) - c e^t$$

$$\underbrace{(c e^t - 1)}_{\neq 0} y(t) = c e^t$$

$$y(t) = \frac{c e^t}{c e^t - 1} \qquad c > 0$$

$$\Leftrightarrow \quad y(t) = \frac{e^t}{e^t - c} \qquad c > 0$$

Condizione: $\boxed{e^t - c \neq 0}$

$$e^t \neq c \quad \Leftrightarrow \quad t \neq \ln(c)$$

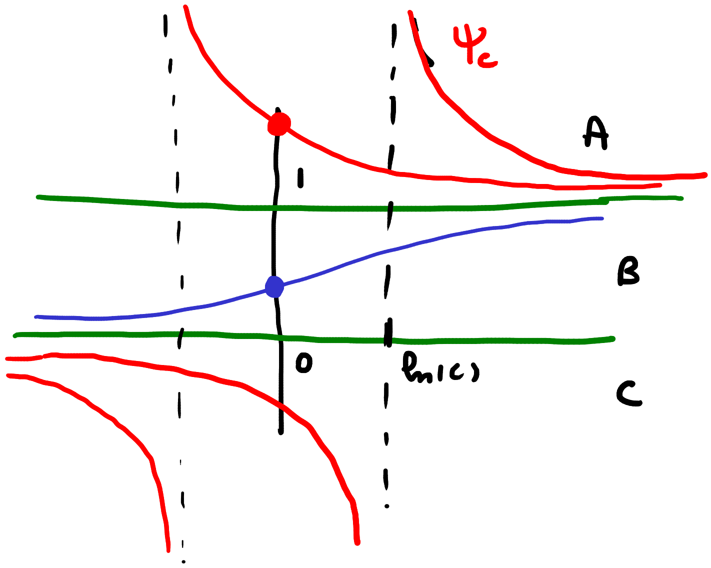
$\forall c > 0$ ho <u>due</u> soluzioni:

$$C \to \varphi_c(t) = \frac{e^t}{\underbrace{e^t - c}_{<0}} \qquad t \in]-\infty, \ln(c)[$$

$$A \to \varphi_c(t) = \frac{e^t}{\underbrace{e^t - c}_{>0}} \qquad t \in]\ln(c), +\infty[$$

$$\lim_{t\to-\infty} \varphi_c(t) = 0 \,, \qquad \lim_{t\to \ln(c)^-} \varphi_c(t) = -\infty$$

$$\lim_{t\to+\infty} \varphi_c(t) = 1 \,, \qquad \lim_{t\to \ln(c)^+} \varphi_c(t) = +\infty$$



$y = y(t)$

$$y' = \underset{\substack{\uparrow\\ >0}}{a}\,y - \underset{\substack{\uparrow\\ b>0}}{b}\,y = (\underbrace{a-b}_{c})y$$

(Malthus) $\begin{cases} y' = cy \\ y(0) = y_0 \end{cases}$ $\qquad y(t) = y_0 e^{ct}$

(Verhulst) $\qquad y' = \cancel{c}\,y\,(\cancel{k} - y)$ costante, 1

carrying capacity

capacità portante

Osservazione ("teor. dell'asintoto")

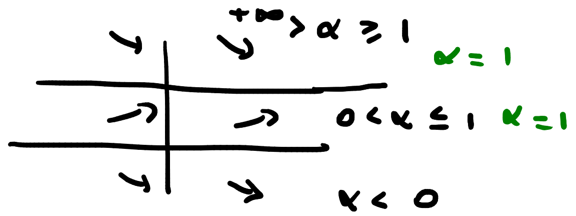
$\varphi : [a, +\infty[\to \mathbb{R}$, φ derivabile

Supp. $\exists \lim_{t\to+\infty} \varphi(t) =: \alpha \ (\in \overline{\mathbb{R}})$

$\exists \lim_{t\to+\infty} \varphi'(t) =: \beta \ (\in \overline{\mathbb{R}})$

Analogo per $\varphi :]-\infty, a] \to \mathbb{R}$ e limiti per $t \to -\infty$.

Se $\alpha \in \mathbb{R}$, allora: $\beta = 0$



Torno a $y' = y(1-y)$

Supp. che $\varphi : [a, +\infty[\longrightarrow \mathbb{R}$ sia soluzione.

Per monotonia: $\exists \lim_{t \to +\infty} \varphi(t) =: \alpha$

Suppongo $\alpha \in \mathbb{R}$

$$\forall t: \quad \varphi'(t) = \underbrace{\varphi(t)}_{\to \alpha} \, (1 - \underbrace{\varphi(t)}_{\to 1-\alpha}) \xrightarrow[t \to +\infty]{} \alpha(1-\alpha) =: \beta$$

"Teor. dell' asint." $\Rightarrow \beta = 0 \iff \alpha(1-\alpha) = 0$

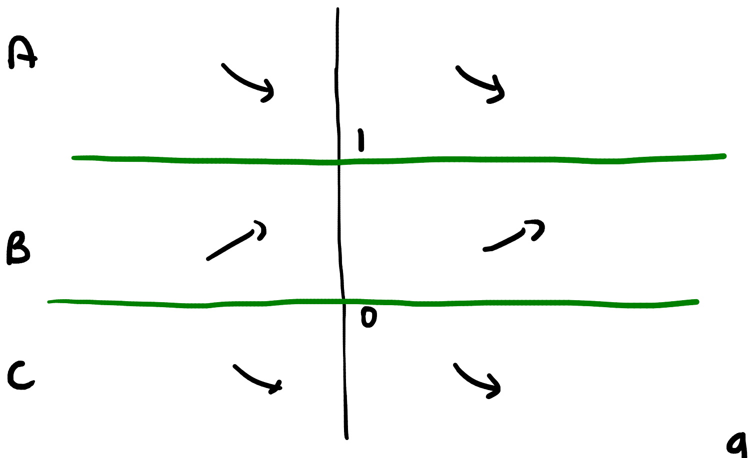
$\iff \alpha = 0$ opp. $\alpha = 1$

$$y' = \frac{y(1-y)}{t^2+1} \quad (1)$$

$g(t) = \frac{1}{t^2+1}$ $\quad g : \mathbb{R} \to \mathbb{R}$ cont.

$h(y) = y(1-y)$ quella di prima

$\Rightarrow$ TEUL applicabile per ogni: $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$



Stesse sol. di equilibrio dell' eq. logistica

$g(t) > 0 \quad \forall t \in \mathbb{R} \Rightarrow$
la monotonia delle sol. è la stessa dell' eq. log.

Asintoti?

Suppongo $\varphi: [a,+\infty[\to \mathbb{R}$ soluz. di (1)

Per la monotonia: $\exists \lim_{t\to+\infty} \varphi(t) =: \alpha$

Suppongo $\underline{\alpha \in \mathbb{R}}$; valuto

$$\lim_{t\to+\infty} \varphi'(t) = \lim_{t\to+\infty} \frac{\overset{\to\alpha}{\varphi(t)}\,\overbrace{(1-\varphi(t))}^{\to 1-\alpha}}{\underbrace{t^2+1}_{\to+\infty}} = 0$$ per qualsiasi α

$\Rightarrow$ in questo caso, il "teor. dell' asintoto" non dà alcuna informazione!

Cerco soluzioni non costanti.

Cerco H primitiva di $\frac{1}{h}$

$$H'(y) = \frac{1}{y(1-y)}$$

La conosco già: $H(y) = \ln\left|\frac{y}{1-y}\right|$

Cerco G primitiva di g:

$$G'(t) = \frac{1}{t^2+1}$$

Scelgo $G(t) = \arctan t$

Quindi: $y = y(t)$ è sol. di (1) $\iff$

$$H(y(t)) = G(t) + c', \qquad c' \in \mathbb{R}$$

$$\ln\left|\frac{y(t)}{1-y(t)}\right| = \arctan t + c' \qquad c' \in \mathbb{R}$$

$$\left|\frac{y(t)}{1-y(t)}\right| = e^{\arctan t} \cdot \underbrace{e^{c'}}_{c}$$

$$(*) \quad \left| \frac{y(t)}{1-y(t)} \right| = c\, e^{\arctan t} \qquad c > 0$$

Per le sol. che "vivono in B" (cioè con $0 < y(t) < 1$)

$$(*) \iff \frac{y(t)}{1-y(t)} = c\, e^{\arctan t}$$

$$y(t) = c e^{\arctan t} - c e^{\arctan t}\, y(t)$$

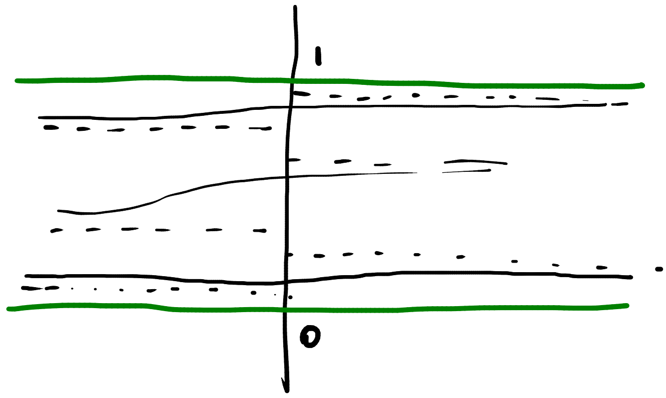
$$(\underbrace{1 + c e^{\arctan t}}_{>0 \ \forall t \in \mathbb{R}})\, y(t) = c\, e^{\arctan t}$$

$$\boxed{y_c(t) = \frac{c e^{\arctan t}}{1 + c e^{\arctan t}}, \qquad t \in \mathbb{R}} \qquad \boxed{c>0}$$

$\lim_{t \to +\infty} y_c(t)$

$$\text{"}\, y_c(+\infty) \text{"} = \frac{c e^{\pi/2}}{1 + c e^{\pi/2}}$$

$$y_c(-\infty) = \frac{c e^{-\pi/2}}{1 + c e^{-\pi/2}}$$



Per le sol. che "vivono in A o in C":

$$(*) \iff \frac{y(t)}{y(t)-1} = c e^{\arctan t}$$

$$y(t) = c e^{\arctan t}\, y(t) - c e^{\arctan t}$$

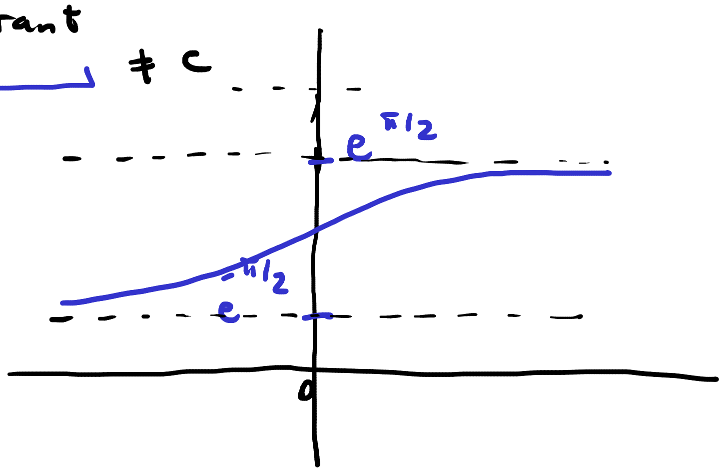
$$(\underbrace{c e^{\arctan t} - 1}_{\neq 0})\, y(t) = c\, e^{\arctan t}$$

$$y(t) = \frac{c e^{\arctan t}}{c e^{\arctan t} - 1}$$

$$\Leftrightarrow \quad y(t) = \frac{e^{\arctan t}}{e^{\arctan t} - c} \qquad c>0$$

Condizione: $\quad e^{\arctan t} - c \neq 0 \quad \odot$

$$\underbrace{e^{\arctan t}}_{?} \neq c$$



se $\quad 0 < c \leq e^{-\pi/2}$:

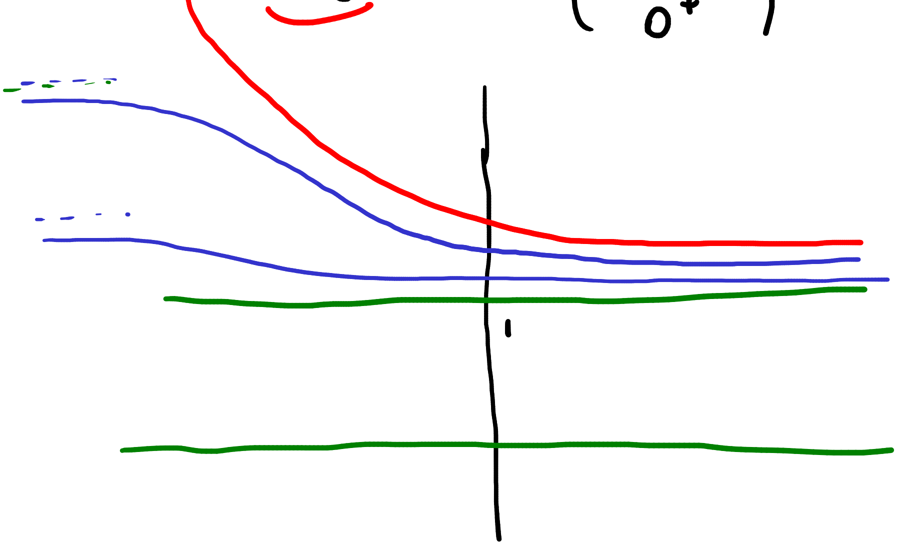
$e^{\arctan t} - c > 0 \quad \forall t \in \mathbb{R}$

$\Rightarrow \odot$ soddisfatta $\forall t \in \mathbb{R}$

$$\Rightarrow \quad \underbrace{\varphi_c(t)}_{>0} = \frac{e^{\arctan t}}{\underbrace{e^{\arctan t} - c}_{>0}} \quad , \quad t \in \mathbb{R}$$

$$\varphi_c(+\infty) = \frac{e^{\pi/2}}{e^{\pi/2} - c}$$

$$\varphi_c(-\infty) = \begin{cases} 0 < c < e^{-\pi/2} & \dfrac{e^{-\pi/2}}{e^{-\pi/2} - c} \\ c = e^{-\pi/2} & \left(\dfrac{e^{-\pi/2}}{0^+}\right) = +\infty \end{cases}$$



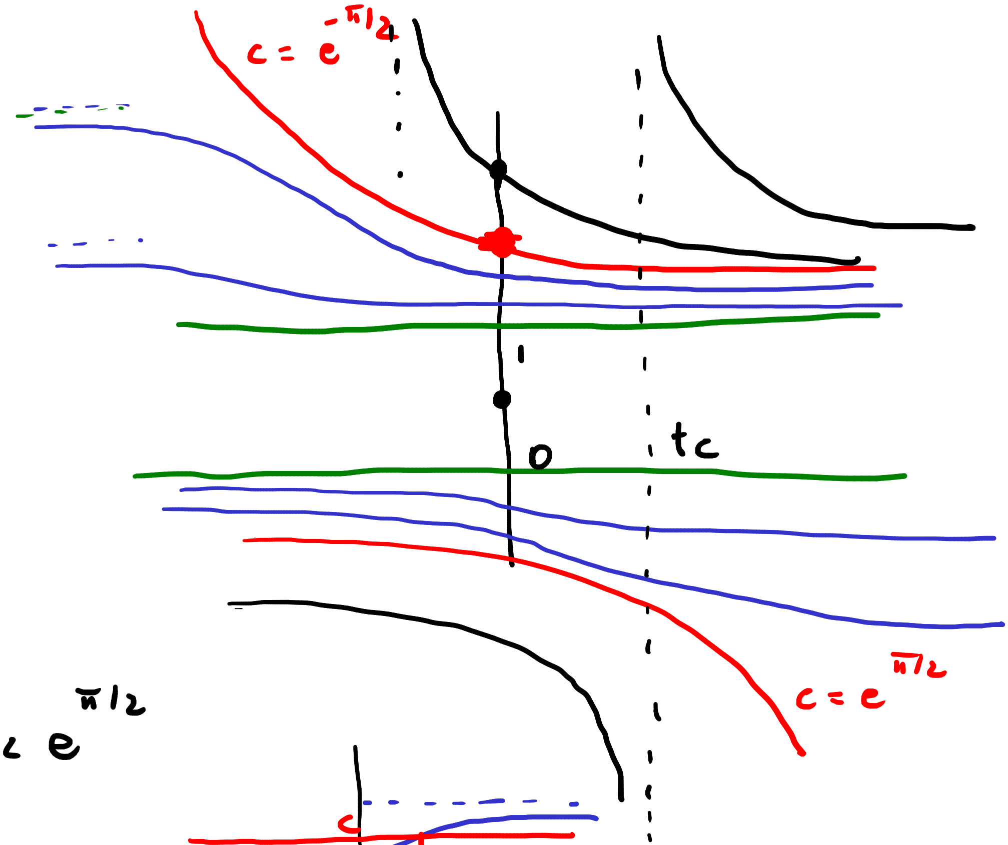
Se $c \geq e^{\pi/2}$:

$$e^{\arctan t} - c < 0 \quad \forall t \in \mathbb{R}$$

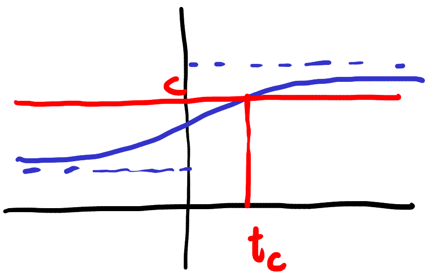
$$\Rightarrow \quad \underbrace{\psi_c(t)}_{<0} = \frac{e^{\arctan t}}{\underbrace{e^{\arctan t} - c}_{<0}} \qquad t \in \mathbb{R}$$

$$\psi_c(-\infty) = \frac{e^{-\pi/2}}{e^{-\pi/2} - c} \qquad \forall c \geq e^{\pi/2}$$

$$\psi_c(+\infty) = \begin{cases} \dfrac{e^{\pi/2}}{e^{\pi/2} - c} \ (\in \mathbb{R}) & c > e^{\pi/2} \\ -\infty & c = e^{\pi/2} \end{cases}$$



$$e^{-\pi/2} < \underbrace{c}_{} < e^{\pi/2}$$



$\exists!\, t_c \in \mathbb{R}$ t.c.

$$e^{\arctan t} - c \begin{cases} < 0 & t < t_c \\ = 0 & t = t_c \\ > 0 & t > t_c \end{cases}$$

Due soluzioni:

$$C' \nearrow \underbrace{\eta_c(t)}_{<0} = \frac{e^{\arctan t}}{e^{\arctan t} - c} \qquad t \in \left]-\infty, t_c\right[$$

$$A \nearrow \underbrace{\xi_c(t)}_{>0} = \ldots\ldots \qquad t \in \left]t_c, +\infty\right[$$

$$\eta_c(-\infty) = \ldots \qquad \eta_c(t_c^-) = -\infty$$

$$\xi_c(+\infty) = \ldots \qquad \xi_c(t_c^+) = +\infty$$

$$y' = \underbrace{\frac{3t^2 - 4t + 3}{2(y-1)}}_{=:\, f(t,y)}$$

$g(t) = 3t^2 - 4t + 3$

$g : \mathbb{R} \to \mathbb{R}$ continua

$g(t) > 0 \quad \forall t \in \mathbb{R}$

$h(y) = \frac{1}{2(y-1)}$

$h : \mathbb{R} \setminus \{1\} \to \mathbb{R}$

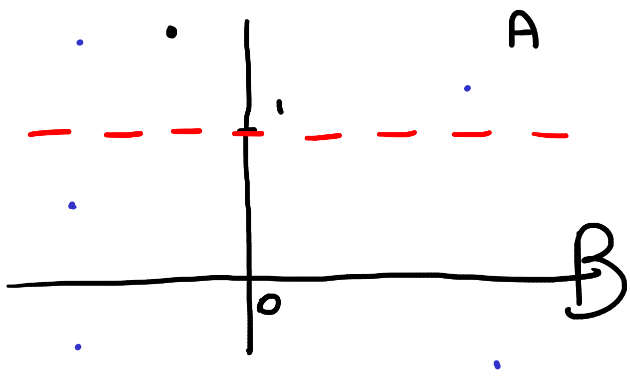
$h \in C^1$

$$f : \overbrace{\underbrace{\mathbb{R} \times \mathbb{R} \setminus \{1\}}}^{\Omega} \to \mathbb{R}$$

(non è di "tipo striscia")

TEUL è applicabile $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R} \setminus \{1\}$

(e grafici di sol. diverse non si intersecano mai)



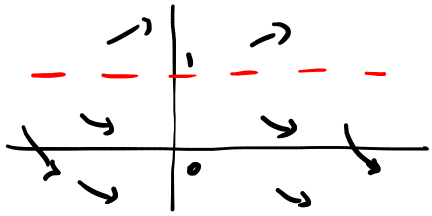
- $h(y) \neq 0 \quad \forall y$

$\Rightarrow$ non ci sono sol. costanti

- ciascuna sol. ha grafico interamente contenuto in $A = \{(t,x) \mid x > 1\}$ oppure in $B = \{(t,x) \mid x < 1\}$

Sulla monotonia delle soluzioni:

$$y' = \frac{\overbrace{3t^2 - 4t + 3}^{>0\ \forall t}}{2(y-1)}$$



Le sol. con valori > 1 sono crescenti;
le sol. con valori < 1 sono decrescenti

Cosa ci dice il "teor. dell' asintoto"?

Suppongo $\varphi : [a, +\infty[\to \mathbb{R}$ soluzione

$\overset{\text{monot.}}{\Rightarrow} \quad \exists \lim_{t \to +\infty} \varphi(t) =: \alpha$

Suppongo $\alpha \in \mathbb{R}$

Calcolo

$$|\varphi'(t)| = \left| \frac{\overbrace{3t^2 - 4t + 3}^{\xrightarrow{t\to+\infty} +\infty}}{\underbrace{2(\varphi(t) - 1)}_{\to 2(\alpha - 1)}} \right| = +\infty \qquad \text{qualunque sia } \alpha \in \mathbb{R}\,!$$

Quindi: nessun $\alpha \in \mathbb{R}$ è ammissibile

$\Rightarrow$ tutte le (eventuali) soluzioni definite in intervalli illimitati superiormente divergono.

Suppongo $\varphi :]-\infty, a] \to \mathbb{R}$ sol. dell' eq.

$\overset{\text{monot}}{\Rightarrow} \quad \exists \lim_{t \to -\infty} \varphi(t) =: \alpha$

Dato che le sol. non possono "attraversare $y=1$", necessariamente $\alpha \in \mathbb{R}$

($\alpha \geq 1$ per sol. in A, $\alpha \leq 1$ per sol. in B)

Calcolo $$|\varphi'(t)| = \left| \frac{\overbrace{3t^2 - 4t + 3}^{\to +\infty}}{2(\varphi(t) - 1)} \right| \xrightarrow[t \to -\infty]{} +\infty$$

assurdo!

Conclusione: questa eq. diff. <u>non ha</u> soluzioni definite in intervalli illimitati inferiormente.

Risolvo l'equazione.

Cerco H primitiva di $\frac{1}{h}$:

$$H'(y) = \frac{1}{h(y)} = 2(y-1)$$

$\Rightarrow$ scelgo $H(y) = (y-1)^2$

Cerco G primitiva di g:

$$G'(t) = 3t^2 - 4t + 3$$

$\Rightarrow$ scelgo $G(t) = t^3 - 2t^2 + 3t$

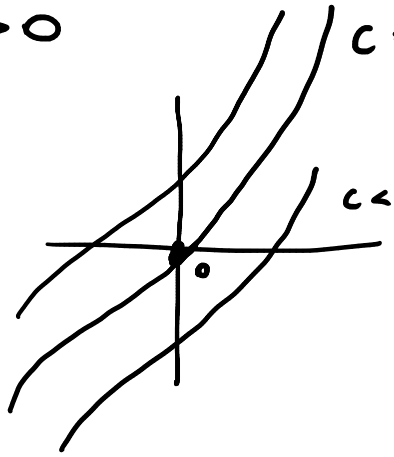
$y = y(t)$ sol. dell'eq:

$$H(y(t)) = G(t) + c, \qquad c \in \mathbb{R}$$

$$\overbrace{(\underbrace{\underbrace{y(t)}_{} - 1}_{\neq 1 \; \neq 0})^2}^{\geq 0} = t^3 - 2t^2 + 3t + c \qquad c \in \mathbb{R}$$

> 0

Condizione: $\overbrace{\underbrace{t^3-2t^2+3t}_{G(t)}+c}^{G_c(t)} > 0$



$$G'(t) = \underbrace{g(t)}_{>0}$$

$\forall c \in \mathbb{R}$: $\exists!\, t_c \in \mathbb{R}$ t.c.

$$G_c(t) \begin{cases} <0 & t<t_c \\ =0 & t=t_c \\ \underline{>0} & \underline{t>t_c} \end{cases}$$

Quindi: $\forall c \in \mathbb{R}$ ci sono <u>due</u> soluzioni:

$$\varphi_c(t) = 1+\sqrt{G_c(t)} = 1+\sqrt{t^3-2t^2+3t+c} \qquad t \in \,]t_c, +\infty[$$

$$\psi_c(t) = 1-\sqrt{G_c(t)} = 1-\sqrt{t^3-2t^2+3t+c}$$

$\varphi_c(+\infty) = +\infty \qquad (t \to +\infty:\ \varphi_c(t) \sim t^{3/2}$)

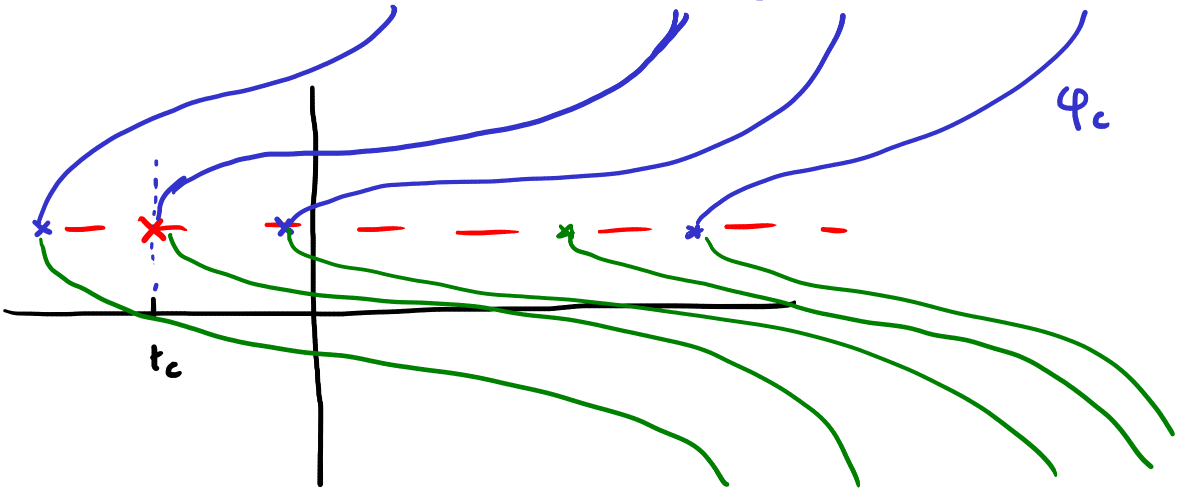
$\psi_c(+\infty) = -\infty \qquad\qquad \psi_c(t) \sim -t^{3/2}$

$t \to t_c^+$: $\varphi_c(t) \to 1$

$\psi_c(t) \to 1$

$$\varphi_c'(t) = \frac{\overbrace{3t^2-4t+3}^{\to\ \text{valore} > 0}}{\underbrace{2(\varphi_c(t)-1)}_{\to\ 0^+}} \longrightarrow +\infty$$

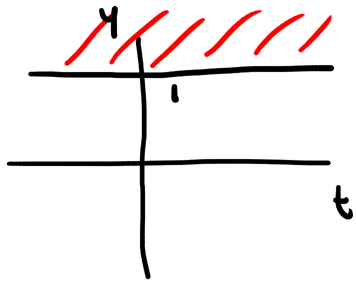
ψ_c' $\qquad -$



$$y' = 2t\sqrt{(1-y)^3} = \underbrace{\underbrace{2t}_{g(t)}\,\underbrace{(1-y)^{3/2}}_{h(y)}}_{:= f(t,y)}$$

$1-y \geq 0$

$y \leq 1$



$g: \mathbb{R} \to \mathbb{R}$ continua

$h:]-\infty, 1] \to \mathbb{R}$ di classe C^1

$\mathrm{dom}(f) = \mathbb{R} \times]-\infty, 1]$ non è aperto

TEUL applicabile in $\mathbb{R} \times]-\infty, 1[$

$\Rightarrow$
- $\forall (t_0, y_0) \in \mathbb{R} \times]-\infty, 1[$: $\exists!$ sol. loc. del PdC corrisp.
- grafici di sol. distinte non si intersecano in alcun punto del tipo $(\bar{t}, \bar{y})$ con $\bar{t} \in \mathbb{R}$, $\bar{y} < 1$.

Oss: $h(y) = 0 \iff y = 1$

$\Rightarrow y(t) \equiv 1$ è sol. costante

$f(t,y) = 2t\sqrt{(1-y)^3} = 0$

$\iff y = 1$ oppure $t = 0$?

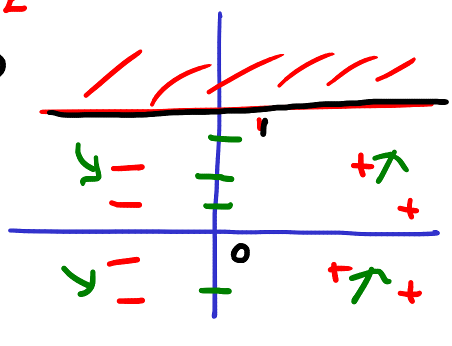
Oss: suppongo $\varphi: I \to \mathbb{R}$ sol dell'eq. diff. con $0 \in I$

Risulta: $\varphi'(0) = f(0, \varphi(0)) = 0$

$t = 0$ "zero-clina"

segno di f

monotonia delle sol.



$$f(t,y) = 2t\,\underbrace{\sqrt{(1-y)^3}}_{\geq 0}$$

DA COMPLETARE ...