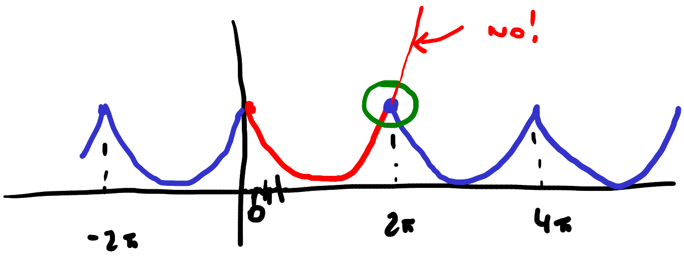
14 novembre 2023

$x \in [0, 2\pi[\longmapsto (x-\pi)^2$ prolung. 2π-periodico



$f|_{]0,2\pi[}$ è continua (è di classe C^1)

$\lim_{x \to 2\pi^-} f(x) = \lim_{x \to 2\pi^-} (x-\pi)^2 = \pi^2 \in \mathbb{R}$

$=$

$\lim_{x \to 2\pi^+} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x-\pi)^2 = \pi^2 \in \mathbb{R}$

$= f(0) = f(2\pi)$

$\Rightarrow$ f è continua in 2π; per periodicità è continua in $\mathbb{R}$.

$\forall x \in]0, 2\pi[: \quad f'(x) = 2(x-\pi)$

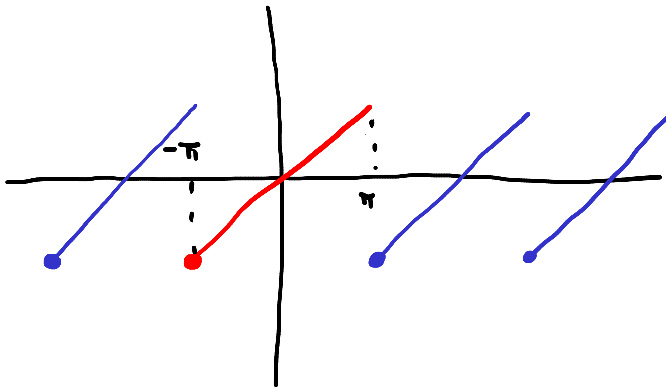
$\Rightarrow \left.\begin{array}{l} \exists \lim_{x \to 0^+} f'(x) = -2\pi \in \mathbb{R} \\ \exists \lim_{x \to 2\pi^-} f'(x) = 2\pi \in \mathbb{R} \end{array}\right| \Rightarrow$ f è reg. a tratti

$\Rightarrow$ la serie di Fourier di f converge a f uniformemente in $\mathbb{R}$.

$\Rightarrow \forall x \in \mathbb{R}: \quad f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2} \cos(nx)$

$\Rightarrow \underset{\pi^2}{\underset{\parallel}{f(0)}} = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2} \quad \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

ora è giustificato!



prolung. 2π - periodico di

$x \in [-\pi, \pi[\longmapsto x$

$f_{|]-\pi,\pi[} \in C^1$

$\lim_{x\to\pi^-} f(x) = \pi \in \mathbb{R}$

$\lim_{x\to-\pi^+} f(x) = -\pi \in \mathbb{R}$ $\Rightarrow$ f è continua a tratti

"

$\lim_{x\to\pi^+} f(x)$

$\Rightarrow$ In π (e traslati periodici) f non è continua

$$\tilde{f}(\pi) = \frac{f(\pi^+) + f(\pi^-)}{2} = \frac{-\pi + \pi}{2} = 0$$

(altrove: $\tilde{f} = f$)

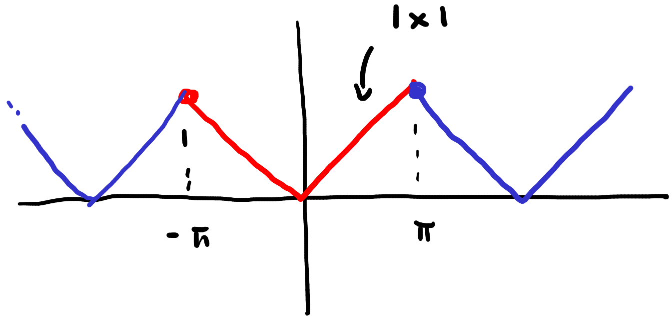
$\lim_{x\to\pi^-} f'(x) = \underbrace{1}_{\in\mathbb{R}} = \lim_{x\to-\pi^+} f'(x)$ $\Rightarrow$ f è regolare a tratti

Conclusione:

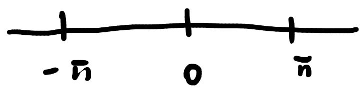
la serie di Fourier di f converge puntualmente in $\mathbb{R}$ a

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus \{(2n+1)\pi \mid n \in \mathbb{Z}\} \\ 0 & x \in \{(2n+1)\pi \mid n \in \mathbb{Z}\} \end{cases}$$

e uniformemente a f in qualsiasi intervallo compatto contenuto in $\mathbb{R} \setminus \{(2n+1)\pi \mid n \in \mathbb{Z}\}$



$$\left.\begin{array}{l} f_{|\,]-\pi,\pi[} \quad \text{continua} \\ \lim_{x\to\pi^-} f(x) = \underset{\substack{\shortparallel \\ f(-\pi)}}{\pi} = \lim_{x\to -\pi^+} f(x) \end{array}\right| \Rightarrow f \text{ è continua in } \mathbb{R}$$



$f_{|\,]-\pi,0[}\ ,\ f_{|\,]0,\pi[}$ di classe C^1

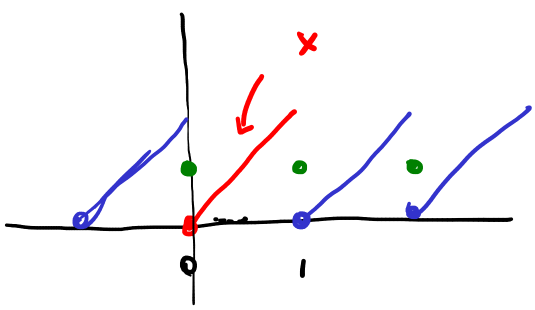
$\forall x \in\,]-\pi,0[\, \cup\,]0,\pi[\,:$

$f'(x) = \operatorname{sgn}(x)$

$$\left.\begin{array}{l} \exists \lim_{x\to\pi^-} f'(x) \\ \exists \lim_{x\to-\pi^+} f'(x) \\ \exists \lim_{x\to 0^+} f'(x) \\ \exists \lim_{x\to 0^-} f'(x) \end{array}\right\} \in \mathbb{R}$$

$\Rightarrow f$ è regolare a tratti.

Quindi: la serie di Fourier di f converge a f uniformemente in $\mathbb{R}$.



- f è continua a tratti
- discontinua nei punti di $\mathbb{Z}$

$$\left.\begin{array}{l} \lim_{x\to 0^+} f(x) = 0 \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 1^-} f(x) = 1 \end{array}\right| \Rightarrow$$

$$\tilde{f}(0) = \frac{0+1}{2} = \frac{1}{2}$$

f è reg. a tratti (...)

Quindi: la serie di Fourier di f converge

- puntualmente in $\mathbb{R}$ a

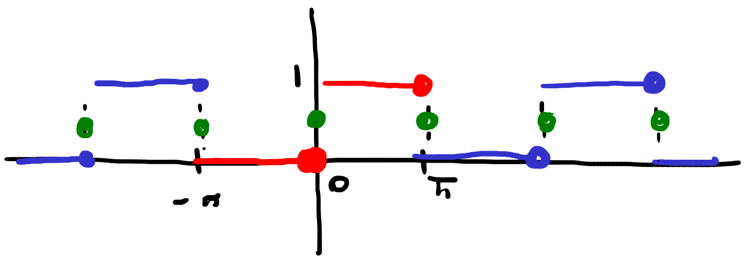
$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus \mathbb{Z} \\ \frac{1}{2} & x \in \mathbb{Z} \end{cases}$$

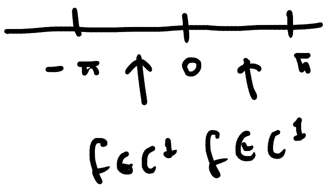
- uniformemente a f in qualsiasi intervallo compatto contenuto in $\mathbb{R} \setminus \mathbb{Z}$.

"Onda quadra":

prolungamento 2π-periodico di

$$x \longmapsto \begin{cases} 0 & x \in]-\pi, 0] \\ 1 & x \in]0, \pi] \end{cases}$$





f reg. a tratti

La serie di Fourier di f converge

- puntualmente in $\mathbb{R}$ a

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\} \\ \frac{1}{2} & x \in \{k\pi \mid k \in \mathbb{Z}\} \end{cases}$$

- uniformemente a f in qualsiasi intervallo compatto contenuto in $\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$.

Calcolo i coefficienti di Fourier:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} \underbrace{f(x)}_{=0}\,dx + \int_{0}^{\pi} \underbrace{f(x)}_{=1}\,dx\right)$$

$$= \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2}.$$

$\forall n \geq 1$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\cos(nx)\,dx = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin(nx)\,dx$$

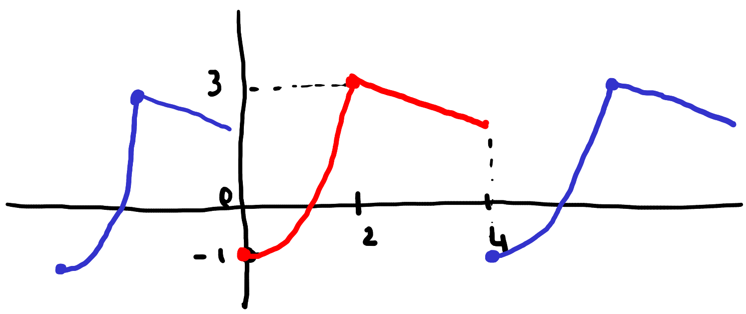
$$= \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{1}{\pi}\,\frac{-(-1)^n+1}{n}$$

$$= \begin{cases} 0 & n \text{ pari} \\ \dfrac{2}{\pi n} & n \text{ dispari} \end{cases}$$

$$\forall x \in \mathbb{R}: \quad \tilde{f}(x) = \frac{1}{2} + \sum_{n=0}^{+\infty} \frac{2}{\pi(2n+1)}\sin((2n+1)x)$$

$$f(x) = \begin{cases} x^2-1 & x \in [0,2[\\ 4-\dfrac{x}{2} & x \in [2,4[\end{cases}$$

prolungamento 4-periodico



$]0,2[$ ✓

$]2,4[$ ✓

$$\lim_{x\to 0^+} f(x) = -1 \quad ; \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 4^-} f(x) = 2$$

$$\neq$$

f non è continua in $x=0$ (e multipli)

... In $x=2$ f è continua

$$\left.\begin{array}{l}
\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 2x = 0 \in \mathbb{R} \\
\lim_{x\to 4^-} f'(x) = \lim_{x\to 4^-} \left(-\frac{1}{2}\right) \in \mathbb{R} \\
\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} 2x = 4 \in \mathbb{R} \\
\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+} \left(-\frac{1}{2}\right) \in \mathbb{R}
\end{array}\right] \Rightarrow f \text{ è reg. a tratti}$$

Quindi: la serie di Fourier di f converge

- puntualm. in $\mathbb{R}$ a $\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus \{4k \mid k \in \mathbb{Z}\} \\ \frac{1}{2} & x \in \{4k \mid k \in \mathbb{Z}\} \end{cases}$

- uniformemente a f negli int. compatti di $\mathbb{R} \setminus \{4k \mid k \in \mathbb{Z}\}$

Calcolo i coefficienti di Fourier:

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{4}\int_0^4 f(x)\,dx$$

$$= \frac{1}{4}\left(\int_0^2 (x^2-1)\,dx + \int_2^4 \left(4-\frac{x}{2}\right)dx\right) = \dots$$

$\forall n \geq 1$:

$$a_n = \frac{1}{\frac{T}{2}} \int_0^T f(x) \cos\left(\frac{2\pi}{T} n x\right) dx$$

$$= \frac{1}{2} \left(\int_0^2 (x^2-1) \cos\left(\frac{\pi}{2} n x\right) dx + \int_2^4 \left(4 - \frac{x}{2}\right) \cos\left(\frac{\pi}{2} n x\right) dx \right)$$

$$= \dots$$

$$b_n = \frac{1}{2} \left(\int_0^2 (x^2-1) \sin\left(\frac{\pi}{2} n x\right) dx + \int_2^4 \left(4 - \frac{x}{2}\right) \sin\left(\frac{\pi}{2} n x\right) dx \right)$$

$$= \dots$$

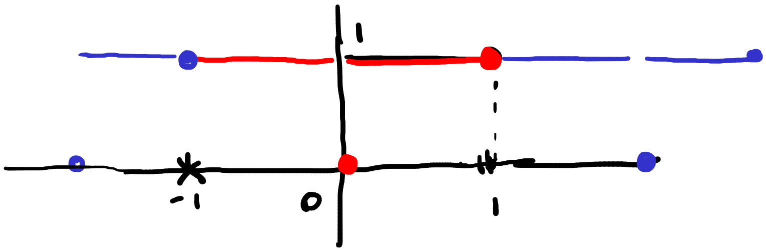
$\forall x \in \mathbb{R}$:

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{+\infty} \left(a_n \cos\left(\frac{\pi}{2} n x\right) + b_n \sin\left(\frac{\pi}{2} n x\right) \right) \quad \square$$

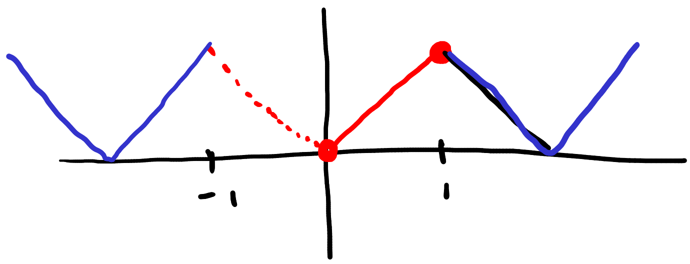
f pari, 2-periodica

$f_\alpha(0) = 0, \qquad f_\alpha(x) = x^\alpha \quad \forall x \in]0,1]$

$\alpha = 0$:

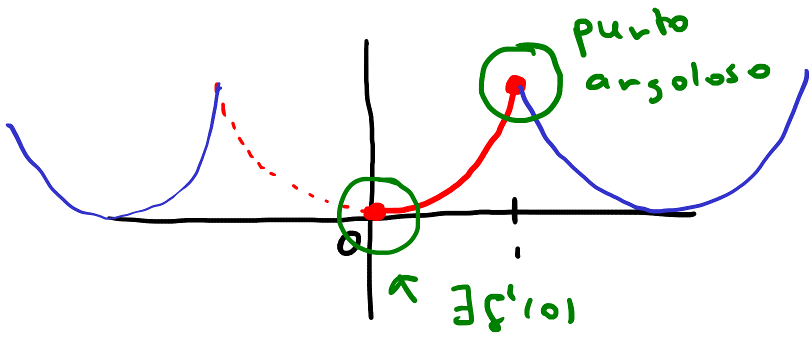


$\alpha = 1$



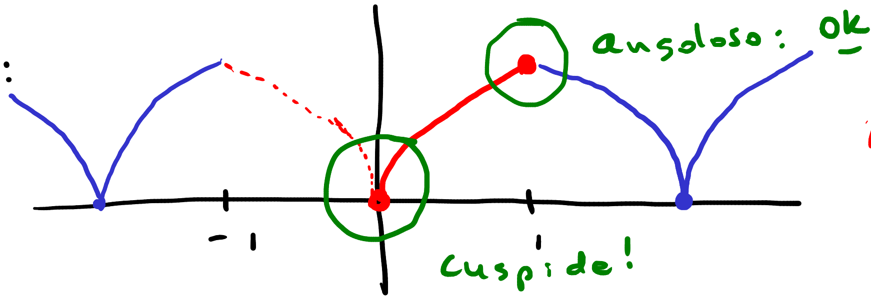
già vista!
(con periodo 2π)

$\alpha > 1$:



Teor: ✓

$0 < \alpha < 1$:

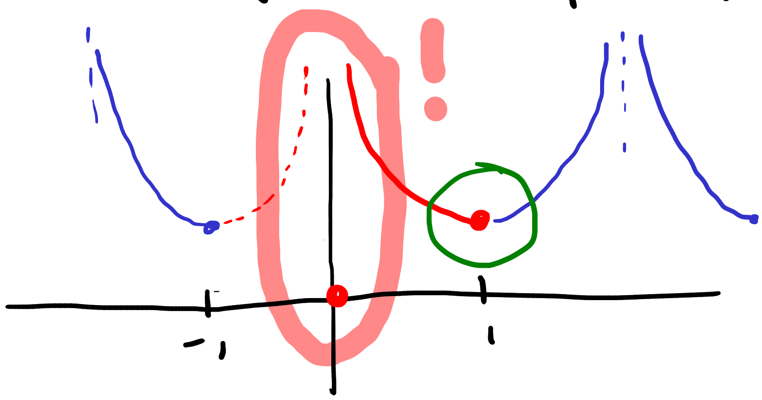


Non regol, a tratti!

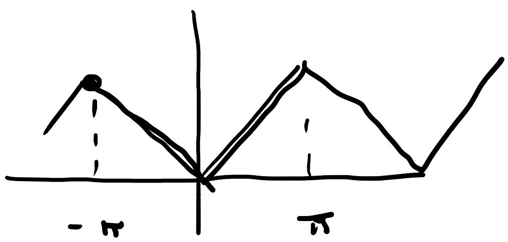
$$\left.\begin{array}{l} f \text{ cont} + f \text{ monotona} \\ \text{a tratti} \end{array}\right) \Rightarrow$$ la s. di F. converge puntualm. a f in $\mathbb{R}$

Sulla conv. unif: non possiamo dire nulla.

$\alpha < 0$:



f non è continua a tratti $\Rightarrow$ il teor. non si applica $\square$



$$\underbrace{\frac{\pi}{2} + \sum_{n=0}^{+\infty} \frac{4}{\pi(2n+1)^2} \cos((2n+1)x)} = f(x) \quad \forall x \in \mathbb{R}$$

Per $x = 0$:

$$0 = \frac{\pi}{2} - \sum_{n=0}^{+\infty} \frac{4}{\pi(2n+1)^2}$$

$$\Rightarrow \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

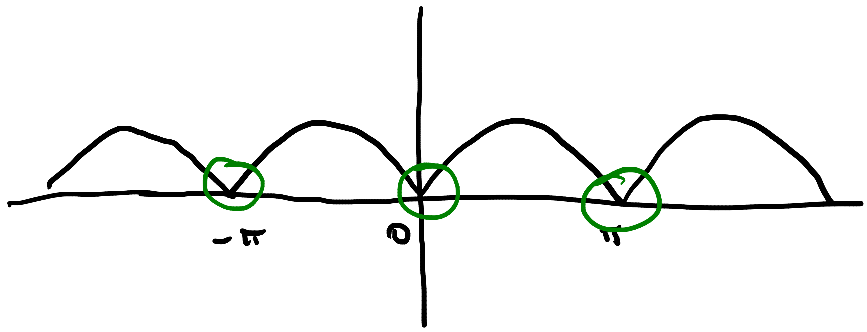
Già ricordato: $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{8} = \frac{\pi^2}{24}.$$

$f(x) = |\sin(x)|$

$\uparrow$

ha periodo minimo $\underline{\pi}$



Teor: $\underline{ok}$

Coefficienti: $b_n = 0 \quad \forall n \geq 1$ (f pari)

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} \sin(x)\, dx = \frac{2}{\pi}$$

$$a_n = \frac{1}{\frac{\pi}{2}} \int_0^{\pi} f(x) \cos\left(\frac{2\pi}{\pi} n x\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(2nx)\, dx = \dots$$

$F: \mathbb{R}^3 \to \mathbb{R}$ t.c. $F(t, x_1, x_2) = t^3 - x_1 + \cos(x_2)$

EDO associata:

$$F(t, y, y') = 0$$

$$\| \quad t^3 - y + \cos(y') = 0 \quad \|$$

$F: \mathbb{R}^4 \to \mathbb{R}$ t.c. $F(t, x_1, x_2, x_3) = x_1 + x_3$

EDO associata: $y + y'' = 0$

$y^2 + (y')^2 - 1 = 0$ $\quad F(t, x_1, x_2) = x_1^2 + x_2^2 - 1$

$F: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$

? $(\sin, \mathbb{R})$ è soluzione ?

intervallo ✓

derivabile una volta ✓

$\forall t \in \mathbb{R}: \quad (t, \sin(t), \sin'(t)) \in \mathbb{R}^3$ ovvio!

$\forall t \in \mathbb{R}: \quad F(t, \sin(t), \sin'(t)) =$

$$(\sin(t))^2 + (\sin'(t))^2 - 1 \stackrel{?}{=}$$

$$\sin(t)^2 + \cos(t)^2 - 1 = 0 \quad ✓$$

Oss: se F non dipende esplicitamente da t, l'EDO associata si dice autonoma.

Oss: se (φ, I) è sol. dell'eq. autonoma associata a F, allora:

$\forall c \in \mathbb{R}^*$: (ψ_c, I_c) è soluzione della stessa equazione, dove :

$$I_c := \{ t \in \mathbb{R} \mid t + c \in I \}$$

$$\psi_c : I_c \to \mathbb{R} \quad \text{tc} \quad \psi_c(t) = \varphi(t+c)$$

<u>Es</u> : $y^2 + (y')^2 - 1 = 0$

$\varphi(t) = \sin(t)$, $t \in \mathbb{R}$

$\psi_c(t) = \sin(t+c)$ $t \in \mathbb{R}_c = \mathbb{R}$

φ soluzione :

$\forall t \in \mathbb{R}$: $\varphi(t)^2 + (\varphi'(t))^2 - 1 = 0$

$$\Rightarrow \forall t \in \mathbb{R} \quad (\underbrace{\varphi(t+c)}_{\psi_c(t)})^2 + (\underbrace{\varphi'(t+c)}_{\psi_c'(t)})^2 - 1 = 0$$

$$\Leftrightarrow \forall t \in \mathbb{R} \quad (\psi_c(t))^2 + (\psi_c'(t))^2 - 1 = 0$$

$\Leftrightarrow$ $(\psi_c, \mathbb{R})$ risolve l'eq.