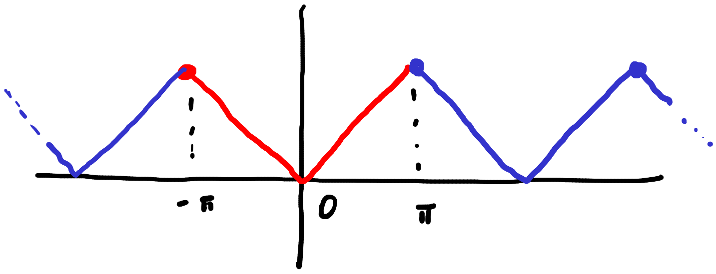
9 novembre 2023

$f: \mathbb{R} \to \mathbb{R}$ prolungamento 2π-periodico di:

$$x \in [-\pi, \pi[\longmapsto |x|$$



f pari $\Rightarrow$ $b_n = 0 \quad \forall n \geq 1$

$$a_0 = \frac{2}{2\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}$$

$\forall n \geq 1$:

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\underbrace{\left[x\,\frac{\sin(nx)}{n}\right]_0^\pi}_{=0} - \int_0^\pi \frac{\sin(nx)}{n}\,dx\right)$$

$$= \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^\pi = \frac{2}{\pi}\,\frac{\cos(n\pi) - \cos(0)}{n^2}$$

$$= \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2} = \begin{cases} 0 & n \text{ pari} \\ -\dfrac{4}{\pi n^2} & n \text{ dispari} \end{cases}$$

Serie di Fourier di f:

$$\frac{\pi}{2} + \sum_{n=0}^{+\infty}\left(-\frac{4}{\pi(2n+1)^2}\right)\overset{\cos((2n+1)x)}{\cancel{\cos(nx)}}$$

$$F(x) := \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}\overset{\cos((2n+1)x)}{\cancel{\cos(nx)}}$$

converge totalmente in $\mathbb{R}$

$\uparrow$

$= f$??

Polinomio e serie di Fourier per funzioni di periodo arbitrario T (>0)

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{1}{T}\int_0^T f(x)\,dx$$

$n \geq 1$: $$a_n = \frac{1}{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2\pi}{T}nx\right)dx$$

$$= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\left(\frac{2\pi}{T}nx\right)dx = \frac{2}{T}\int_0^T \dots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\left(\frac{2\pi}{T}nx\right)dx = \frac{2}{T}\int_0^T \dots$$

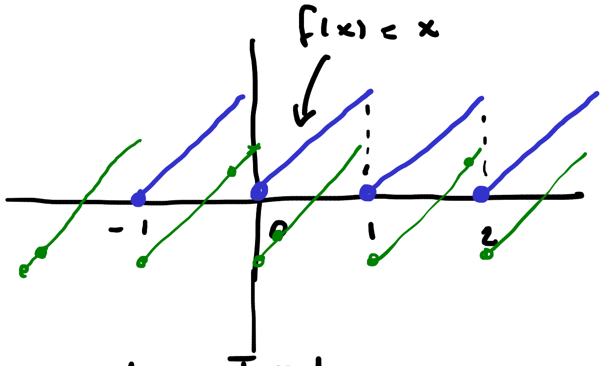
$$F_n(x) = a_0 + \sum_{k=1}^{n}\left(a_k\cos\left(\frac{2\pi}{T}kx\right) + b_k\sin\left(\frac{2\pi}{T}kx\right)\right)$$

↑ polinomio di Fourier

serie di Fourier $$a_0 + \sum_{n=1}^{+\infty}\left(a_n\cos\left(\frac{2\pi}{T}nx\right) + b_k\sin\left(\frac{2\pi}{T}nx\right)\right)$$

f : mantissa

$(f(x) := x - \lfloor x \rfloor)$



$f(x) - \frac{1}{2}$

↑

"dispari"

f è periodica di periodo $T = 1$

$$a_0 = \frac{1}{1}\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

$\forall n \geq 1$:

$$a_n = \frac{1}{\frac{1}{2}}\int_0^1 f(x)\cos\left(\frac{2\pi}{1}nx\right)dx = 2\int_0^1 x\cos(2\pi nx)\,dx$$

$$= 2\left(\underbrace{\left[x \frac{\sin(2\pi n x)}{2\pi n} \right]_0^1}_{=0} - \int_0^1 \frac{\sin(2\pi n x)}{2\pi n}\, dx \right)$$

$$= 2 \left[\frac{\cos(2\pi n x)}{(2\pi n)^2} \right]_0^1 = 0$$

$$b_n = 2\int_0^1 x \sin(2\pi n x)\, dx$$

$$= 2\left(\left[-\frac{x\cos(2\pi n x)}{2\pi n} \right]_0^1 + \int_0^1 \frac{\cos(2\pi n x)}{2\pi n}\, dx \right)$$

$$\left[\frac{\sin(2\pi n x)}{\dots} \right]_0^1 = 0$$

$$= 2\left(-\frac{1}{2\pi n} \right) = -\frac{1}{\pi n}$$

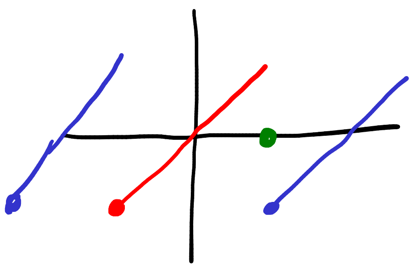
Serie di Fourier di f:

$$\frac{1}{2} + \sum_{n=1}^{+\infty} \left(-\frac{1}{\pi n} \right) \sin(2\pi n x)$$

$$\frac{1}{2} - \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{1}{n} \sin(2\pi n x)$$

converge ?

boh!



$$\sum_{n=1}^{+\infty} \underbrace{b_n}_{\uparrow \; \dots} \sin(nx) =: F(x)$$

$$F(\pi) = 0$$

Dim. del teor. "serie di F. e distanza quadr."

① Definisco

$c_0 = a_0$ | $\varphi_0(x) = 1$
$c_1 = b_1$ | $\varphi_1(x) = \sin(x)$
$c_2 = a_1$ | $\varphi_2(x) = \cos(x)$
$c_3 = b_2$ | $\varphi_3(x) = \sin(2x)$
$c_4 = a_2$ | $\varphi_4(x) = \cos(2x)$
$\vdots$ | $\vdots$

$c_{2n} = a_n$ | $\varphi_{2n}(x) = \cos(nx)$
$c_{2n-1} = b_n$ | $\varphi_{2n-1}(x) = \sin(nx)$

$$\forall n: \quad F_n(x) = a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

$$= \sum_{k=0}^{2n} c_k \varphi_k(x)$$

- $$\langle \varphi_k, \varphi_h \rangle := \int_{-\pi}^{\pi} \varphi_k(x) \varphi_h(x)\, dx = \begin{cases} 0 & k \neq h \\ 2\pi & k = h = 0 \\ \pi & k = h \neq 0 \end{cases}$$

- $\forall m \in \{0, \dots, 2n\}$:

$$\langle F_n, \varphi_m \rangle = \langle \sum_{k=0}^{2n} c_k \varphi_k, \varphi_m \rangle$$

$$= \sum_{k=0}^{2n} c_k \langle \varphi_k, \varphi_m \rangle = c_m \langle \varphi_m, \varphi_m \rangle$$

$$= \begin{cases} c_0 2\pi & m = 0 \\ c_m \pi & m \in \{1, \dots, 2n\} \end{cases}$$

- $\forall m \in \{0, \dots, 2n\}$:

$$\langle f, \varphi_m \rangle = \int_{-\pi}^{\pi} f(x) \varphi_m(x)\, dx = \begin{cases} 2\pi c_0 & m = 0 \\ \pi c_m & m \in \{1, \dots, 2n\} \end{cases}$$

$\Rightarrow \quad \forall m \in \{0, \dots, 2n\}:$

$$\langle f - F_n, \varphi_m \rangle = \langle f, \varphi_m \rangle - \langle F_n, \varphi_m \rangle = 0$$

Quindi: $f - F_n$ è ortogonale a φ_m, $\forall m \in \{0, \dots, 2n\}$

$\Rightarrow$ $f - F_n$ è ortogonale a qualsiasi polinomio trigonometrico di ordine n.

Oss: in un generico spazio vettoriale con prodotto scalare:

$$\underbrace{\|a \pm b\|^2}_{\substack{\text{norma associata} \\ \text{al prod. scalare}}} = \langle a \pm b, a \pm b \rangle = \langle a, a \rangle \pm \langle a, b \rangle + \pm \langle b, a \rangle + \langle b, b \rangle$$

$$= \|a\|^2 + \|b\|^2 \pm 2 \langle a, b \rangle$$

Ora posso dimostrare ①

Sia P_n un arbitrario polinomio trigonometrico di ordine n.

$$\|f - P_n\|^2 = \| \underbrace{f - F_n}_{a} + \underbrace{F_n - P_n}_{b} \|^2$$

$$= \|f - F_n\|^2 + \|F_n - P_n\|^2 + 2 \underbrace{\langle f - F_n, \overbrace{F_n - P_n}^{\text{pol. trig. di ordine } n} \rangle}_{\substack{= 0 \\ \text{(oss. preced.)}}}$$

pol. trig. di ordine n — pol. trig. di ordine n

$$\Rightarrow \|f - P_n\|^2 = \|f - F_n\|^2 + \underbrace{\|F_n - P_n\|^2}_{\geq 0} \geq \|f - F_n\|^2$$

$(= 0 \iff F_n \equiv P_n)$

cioè: $$\int_{-\pi}^{\pi} |f(x) - P_n(x)|^2 \, dx \geq \int_{-\pi}^{\pi} |f(x) - F_n(x)|^2 \, dx \qquad \boxed{1}$$

Dimostro 2

$$\|f - F_n\|^2 = \|f\|^2 + \|F_n\|^2 - 2\langle f, F_n\rangle$$

$$\|F_n\|^2 = \langle F_n, F_n\rangle = \left\langle \sum_{k=0}^{2n} c_k \varphi_k , \sum_{h=0}^{2n} c_h \varphi_h \right\rangle$$

$$= \sum_{k=0}^{2n} c_k \left\langle \varphi_k, \sum_{h=0}^{2n} c_h \varphi_h \right\rangle \underset{\substack{\uparrow \\ \text{ortog.} \\ \text{di } (\varphi_k)}}{=} \sum_{k=0}^{2n} c_k \langle \varphi_k, c_k \varphi_k\rangle$$

$$= \sum_{k=0}^{2n} c_k^2 \langle \varphi_k, \varphi_k\rangle = \underbrace{c_0^2 \, 2\pi + \sum_{k=1}^{2n} c_k^2 \, \pi}$$

$$\langle f, F_n\rangle = \left\langle f, \sum_{k=0}^{2n} c_k \varphi_k \right\rangle = \sum_{k=0}^{2n} c_k \langle f, \varphi_k\rangle$$

$$= c_0 \cdot 2\pi c_0 + \sum_{k=1}^{2n} c_k \pi c_k$$

$$= \underbrace{2\pi c_0^2 + \sum_{k=1}^{2n} \pi c_k^2}$$ sono uguali!

$$\|f - F_n\|^2 = \|f\|^2 + \overbrace{\|F_n\|^2}^{} - 2\overbrace{\langle f, F_n\rangle}^{}$$

$$= \|f\|^2 - \|F_n\|^2$$

$$= \|f\|^2 - \left(2\pi c_0^2 + \sum_{k=1}^{2n} \pi c_k^2\right)$$

Dunque:

$$\underbrace{\int_{-\pi}^{\pi} |f(x) - F_n(x)|^2 \, dx}_{\geq 0} = \int_{-\pi}^{\pi} |f(x)|^2 \, dx - \pi \left(2c_0^2 + \sum_{k=1}^{2n} c_k^2 \right)$$

$$= \underbrace{\int_{-\pi}^{\pi} |f(x)|^2 \, dx - \pi \left(2a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right)}_{\geq 0} \qquad \boxed{2}$$

Dimostro (3)

Da (2) deduco che per ogni n:

$$\int_{-\pi}^{\pi} |f(x)|^2 \, dx - \pi \left(2a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right) \geq 0$$

quindi:

$$\forall n: \quad 2a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

somma parziale n-esima della serie di termine $a_k^2 + b_k^2 \geq 0$

$\Rightarrow$ ha limite per $n \to +\infty$

permanenza disug. $\Rightarrow$

$$2a_0^2 + \sum_{k=1}^{+\infty} (a_k^2 + b_k^2) \leq \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx}_{\in \mathbb{R}} \qquad \boxed{3}$$

Dimostro (4):

(3) $\Rightarrow \sum_n (a_n^2 + b_n^2)$ converge

$$\Rightarrow \underbrace{a_n^2}_{\geq 0} + \underbrace{b_n^2}_{\geq 0} \to 0 \Rightarrow a_n^2 \to 0, \; b_n^2 \to 0$$

$$\Rightarrow a_n \to 0, \; b_n \to 0$$

$\square$

base canonica

$$\left[\; x = \underbrace{(x\cdot e_1)}_{\substack{\text{prod. scalare}\\ \text{in } \mathbb{R}^3}} e_1 + (x\cdot e_2)\, e_2 + (x \cdot e_3)\, e_3 \;\right]$$

nel senso della dist. quadratica

$$f = \text{somma della serie di Fourier di } f$$

$$= \sum_{n=0}^{+\infty} c_n \varphi_n$$

$$= c_0 \varphi_0 + \sum_{n=1}^{+\infty} c_n \varphi_n$$

$$= \frac{\langle f, \varphi_0 \rangle}{2\pi} \varphi_0 + \sum_{n=1}^{+\infty} \frac{\langle f, \varphi_n \rangle}{\pi} \varphi_n$$

$$= \left\langle f, \frac{\varphi_0}{\sqrt{2\pi}} \right\rangle \frac{\varphi_0}{\sqrt{2\pi}} + \sum_{n=1}^{+\infty} \left\langle f, \frac{\varphi_n}{\sqrt{\pi}} \right\rangle \frac{\varphi_n}{\sqrt{\pi}}$$

$$\tilde{\varphi}_n = \begin{cases} \dfrac{\varphi_0}{\sqrt{2\pi}} & n = 0 \\[2ex] \dfrac{\varphi_n}{\sqrt{\pi}} & n \geq 1 \end{cases}$$

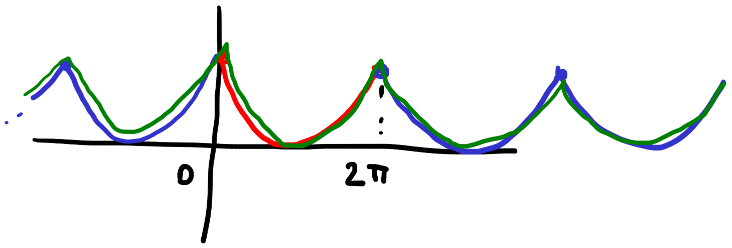
$$= \langle f, \tilde{\varphi}_0 \rangle \tilde{\varphi}_0 + \sum_{n=1}^{+\infty} \langle f, \tilde{\varphi}_n \rangle \tilde{\varphi}_n$$

$$= \sum_{n=0}^{+\infty} \langle f, \tilde{\varphi}_n \rangle \tilde{\varphi}_n$$

Oss: $\langle \tilde{\varphi}_n, \tilde{\varphi}_m \rangle = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

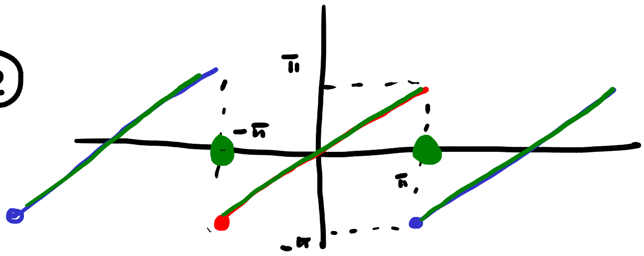
$\{ \tilde{\varphi}_n \}$ sistema ORTONORMALE

1



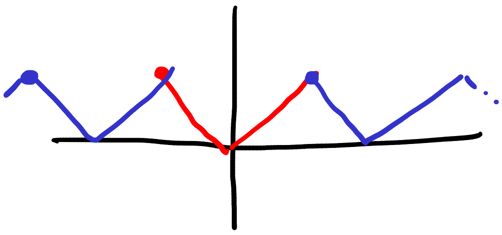
f continua $\Rightarrow \tilde{f} = f$

2



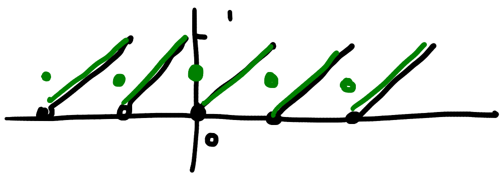
$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus \{(2n+1)\pi \mid n \in \mathbb{Z}\} \\ 0 & x \in \{(2n+1)\pi \mid n \in \mathbb{Z}\} \end{cases}$$

3



$\tilde{f} = f$

4



$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbb{R} \setminus \mathbb{Z} \\ \frac{1}{2} & x \in \mathbb{Z} \end{cases}$$

Dimostro il punto 2 del teor. sulla convergenza delle serie di Fourier

Dato che f è regolare a tratti, posso applicare 1: la serie di Fourier di f converge puntualmente in $\mathbb{R}$ a $\tilde{f}$, che coincide con f (perché f è continua).

M: basta dimostrare che la serie di Fourier di f converge totalmente in $\mathbb{R}$ (e quindi unif.)

Quindi: mi basta dimostrare che le serie di termine a_n e b_n convergono assolutamente.

Osservo che f' (eventualmente prolungata assegnando il valore 0 nei punti in cui non è definita) è una funzione continua a tratti, e quindi per f' vale la disuguaglianza di Bessel.

Perciò: se denoto con a'_n e b'_n i coefficienti di Fourier di f', risulta:

(*) la serie di termine $(a'_n)^2 + (b'_n)^2$ è convergente.

Calcolo a'_n e b'_n; per semplicità suppongo $T = 2\pi$; suppongo anche f' continua in $[-\pi, \pi]$ (in modo da non dover "spezzare" l'integrale tra $-\pi$ e π in corrispondenza dei punti della decomposizione prevista dalla definizione di funzione regolare a tratti).

Per $n \geq 1$:

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\Big[\underbrace{f(x)\cos(nx)}_{=0}\Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-n\sin(nx))\,dx\right)$$

(per periodicità)

$$= n\cdot\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}_{b_n} = n\,b_n$$

$$\Rightarrow a_n' = n\,b_n$$

Analogamente: $b_n' = -n\,a_n$

Pertanto:

$$(a_n')^2 + (b_n')^2 = n^2 b_n^2 + n^2 a_n^2 ,$$

e ⊛ significa che converge la serie di termine $n^2 b_n^2 + n^2 a_n^2$.

Per confronto:

⊛⊛ le serie di termine $n^2 a_n^2$ e $n^2 b_n^2$ convergono.

Osservo che per ogni $n \geq 1$:

$$|a_n| = \underbrace{\frac{1}{n}}_{a}\cdot\underbrace{n|a_n|}_{b} \leq \frac{1}{2}\left(\frac{1}{n^2} + n^2 a_n^2\right)$$

$$\leq \frac{a^2+b^2}{2}$$

$\sum_n \frac{1}{n^2}$ conv. (arm. gen.)

$\sum_n n^2 a_n^2$ conv. $(*)$

$\Rightarrow \sum_n \frac{1}{2}\left(\frac{1}{n^2} + n^2 a_n^2\right)$ converge

somma e multiplo

confronto

$\Rightarrow \sum_n |a_n|$ converge

Analogamente si prova che $\sum_n |b_n|$ converge.

$\square$