31 ottobre 2023

$\{a_n\}, \{b_n\} \subseteq \mathbb{R}$

Definisco $c_n := \sum_{k=0}^{n} a_k b_{n-k}$

$(c_0 = a_0 b_0, \quad c_1 = a_0 b_1 + a_1 b_0, \quad c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0, \dots)$

$\sum_n c_n$ si chiama serie prodotto secondo Cauchy delle serie $\sum_n a_n$ e $\sum_n b_n$.

Teor. (di Mertens)

Se le serie di termine a_n e b_n convergono, e almeno una delle due converge assolutamente, allora:

- la serie di termine c_n converge
- $\sum_{n=0}^{+\infty} c_n = \left(\sum_{n=0}^{+\infty} a_n \right) \left(\sum_{n=0}^{+\infty} b_n \right)$

Torno alle serie di potenze:

$$\sum_{n=0}^{+\infty} a_n (x-x_0)^n =: f(x) \qquad x \in \,]x_0 - R_a, x_0 + \underbrace{R_a}_{>0}[$$

$$\sum_{n=0}^{+\infty} b_n (x-x_0)^n =: g(x) \qquad x \in \,]x_0 - R_b, x_0 + \underbrace{R_b}_{>0}[$$

Considero $R := \min\{R_a, R_b\}$; osservo che entrambe le serie convergono assolutamente in $]x_0 - R, x_0 + R[$

Applico il teor. di Mertens: $\forall x \in \,]x_0 - R, x_0 + R[$

? la serie prodotto secondo Cauchy converge in x e ha somma $f(x)\cdot g(x)$

Il termine n-esimo della serie prodotto secondo Cauchy è

$$\sum_{k=0}^{n} a_k (x-x_0)^k \, b_{n-k} (x-x_0)^{n-k} =$$

$$\sum_{k=0}^{n} a_k b_{n-k} (x-x_0)^{k+n-k} = \left(\sum_{k=0}^{n} a_k b_{n-k}\right)(x-x_0)^n$$

Quindi: la serie p.s.C. delle due serie di potenze è una serie di potenze

- di centro x_0,
- di coefficienti uguali ai "prodotti secondo Cauchy" dei coefficienti delle due serie,
- raggio di convergenza maggiore o uguale al minimo dei r.d.c. delle due serie.

Nell'esempio:

$$y'' + \frac{t}{t^2+1} y' - \frac{1}{t^2+1} y = 0 \qquad y(t) = \sum_{n=0}^{+\infty} c_n t^n$$

$$\sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2} + \underbrace{\left(\sum_{n=0}^{+\infty} (-1)^n t^{2n+1}\right)\left(\sum_{n=1}^{+\infty} n c_n t^{n-1}\right)}_{= \sum_{n=0}^{+\infty} c'_n t^n} +$$

$$- \underbrace{\left(\sum_{n=0}^{+\infty} (-1)^n t^{2n}\right)\left(\sum_{n=0}^{+\infty} c_n t^n\right)}_{\sum_{n=0}^{+\infty} c''_n t^n} = 0$$

Non conviene procedere così (c'_n, c''_n sono "complicati" da determinare).

Procedo direttamente su

$$(t^2+1)\,y'' + t\,y' - y = 0$$

Derivando e sostituendo:

$\forall t \in \,]-1,1[\,$:

$$(t^2+1)\sum_{n=2}^{+\infty} n(n-1)c_n t^{n-2} + t\sum_{n=1}^{+\infty} n c_n t^{n-1} - \sum_{n=0}^{+\infty} c_n t^n = 0$$

$$\Rightarrow \sum_{n=2}^{+\infty} n(n-1)c_n t^n + \sum_{n=2}^{+\infty} n(n-1)c_n t^{n-2} + \sum_{n=1}^{+\infty} n c_n t^n - \sum_{n=0}^{+\infty} c_n t^n = 0$$

$$\Rightarrow \sum_{n=2}^{+\infty} n(n-1)c_n t^n + \sum_{n=0}^{+\infty} (n+2)(n+1)c_{n+2} t^n + \sum_{n=1}^{+\infty} n c_n t^n - \sum_{n=0}^{+\infty} c_n t^n$$

$$= 0 = \sum_{n=0}^{\infty} 0 \cdot t^n$$

P.I.S.P.

$\Rightarrow$

$n=0$: $\quad 2c_2 - c_0 = 0 \quad \Rightarrow \quad c_2 = \dfrac{c_0}{2}$

$n=1$: $\quad 6c_3 + c_1 - c_1 = 0 \quad \Rightarrow \quad c_3 = 0$

$n \geq 2$: $\quad n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_n = 0$

$$(n+2)(n+1)c_{n+2} = c_n - \cancel{nc_n} - n^2 c_n + \cancel{n}c_n$$

$$c_{n+2} = -\frac{(n^2-1)\,c_n}{(n+2)(n+1)} = -\frac{n-1}{n+2}\,c_n$$

c_0 arbitrario; c_1 arbitrario

$c_2 = \dfrac{c_0}{2}$, $\quad c_3 = 0$, $\quad c_4 = -\dfrac{1}{4}c_2 = -\dfrac{1}{8}c_0$

$c_5 = -\dfrac{2}{5}c_3 = 0 \qquad (c_7 = c_9 = \dots = 0)$

$$c_6 = -\frac{3}{6} c_4 = +\frac{3}{6\cdot 8} c_0 \quad \cdots$$

$$\cdots \quad \forall n \geq 2: \quad c_{2n} = (-1)^{n-1} \frac{(2n-3)(2n-5)\cdots 3\cdot 1}{n!\, 2^n}$$

$\forall t \in]-1,1[$:

$$y(t) = c_0 \Big(\overbrace{1 - \frac{1}{2} t^2 - \frac{1}{8} t^4 + \frac{3}{6\cdot 8} t^6 + \cdots}^{y_1(t)} \Big) + c_1 \underbrace{t}_{y_2(t)} \qquad \square$$

$$g(x) := \cos\left(\frac{2\pi}{T} x\right)$$

$$g(x+T) := \cos\left(\frac{2\pi}{T}(x+T)\right) = \cos\left(\frac{2\pi}{T} x + 2\pi\right)$$

$$= \cos\left(\frac{2\pi}{T} x\right) = g(x)$$

$$\sum_n \underbrace{\frac{n^6}{n!} \cos(nx)}_{f_n(x)} \qquad \begin{matrix}(n \geq 1) \\ x \in \mathbb{R}\end{matrix}$$

Fisso $x \in \mathbb{R}$: $\left.\begin{matrix} |f_n(x)| = \left|\frac{n^6}{n!}\cos(nx)\right| \leq \frac{n^6}{n!} \\ \sum_n \frac{n^6}{n!} \text{ conv. (crit. rapporto)}\end{matrix}\right\} \Rightarrow$

$\forall x$: $\sum_n f_n(x)$ conv. assolut.

a posteriori: superflua!

$$\left.\begin{array}{l} \forall x \in \mathbb{R}: \quad |f_n(x)| \leq \dfrac{n^6}{n!} =: M_n \\ \displaystyle\sum_n M_n \text{ converge} \end{array}\right] \Rightarrow$$

$\sum_n f_n(x)$ conv. totalm. in $\mathbb{R}$.

Oss: $a_0 + \sum_n (\underbrace{a_n \cos(nx) + b_n \sin(nx)}_{f_n(x)})$

$$\begin{aligned} \forall x \in \mathbb{R}: \quad |f_n(x)| &= |a_n \cos(nx) + b_n \sin(nx)| \\ &\leq |a_n \cos(nx)| + |b_n \sin(nx)| \\ &= |a_n| \underbrace{|\cos(nx)|}_{\leq 1} + |b_n| \underbrace{|\sin(nx)|}_{\leq 1} \\ &\leq |a_n| + |b_n| \end{aligned}$$

$\Rightarrow$ se $\sum_n |a_n|$ e $\sum_n |b_n|$ convergono, allora:

$\sum_n f_n$ conv. totalm. in $\mathbb{R}$.

Verifico le formule di ortogonalità per il coseno (le altre si verificano allo stesso modo)

$m = n = 0$: $\displaystyle\int_{-\pi}^{\pi} \cos(0x) \cos(0x)\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi$ ✓

$m = n \neq 0$: $\displaystyle\int_{-\pi}^{\pi} \cos(mx) \cos(mx)\, dx =$

$\displaystyle = \int_{-\pi}^{\pi} \cos^2(mx)\, dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2mx)}{2}\, dx$

$$= \frac{1}{2} \cdot 2\pi + \underbrace{\left[\frac{\sin(2mx)}{2 \cdot 2m} \right]_{-\pi}^{\pi}}_{=0} = \pi \quad \checkmark$$

$m \neq n$

Se $n = 0$: $\displaystyle \int_{-\pi}^{\pi} \cos(mx) \underbrace{\cos(nx)}_{=1} \, dx = \int_{-\pi}^{\pi} \cos(mx) \, dx$

$$= \left[\frac{\sin(mx)}{m} \right]_{-\pi}^{\pi} = 0 \quad \checkmark$$

$m \neq n$, $m, n \neq 0$:

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx =$$

$$= \underbrace{\left[\frac{\sin(mx)}{m} \cos(nx) \right]_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} \frac{\sin(mx)}{m} \big(-n \sin(nx) \big) \, dx$$

$$= \frac{n}{m} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx$$

$$= \frac{n}{m} \left(\underbrace{\left[-\frac{\cos(mx)}{m} \sin(nx) \right]_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} -\frac{\cos(mx)}{m} \, n \cos(nx) \, dx \right)$$

$$= \frac{n}{m} \cdot \frac{n}{m} \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx$$

$$\Rightarrow \left(1 - \frac{n^2}{m^2} \right) \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0$$

$n \neq m \Rightarrow \frac{n^2}{m^2} \neq 1$ $\quad \Rightarrow \displaystyle \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0 \quad \checkmark$

$\Rightarrow 1 - \frac{n^2}{m^2} \neq 0$

$\square$

Motivo la definizione dei coeff. di Fourier.

Suppongo che f sia somma $\underline{\text{uniforme}}$ di una serie trigonometrica:

consente di integrare t.a t. in $[-\pi,\pi]$

$$\forall x \in \mathbb{R}: \quad f(x) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Calcolo:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{+\infty} (a_n\cos(nx) + b_n \sin(nx))\,dx$$

integr. t. a t.

$$= a_0 \cdot 2\pi + \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx))\,dx$$

linearità integr.

$$= a_0 \cdot 2\pi + \sum_{n=1}^{+\infty} \Big(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\,dx}_{=0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\,dx}_{=0}\Big)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = a_0\, 2\pi \quad \Rightarrow \quad \underbrace{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Fisso $m \geq 1$:

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\,dx = \, ?$$

(*) $\left[\forall x: \quad f(x)\cos(mx) = a_0 \cos(mx) + \sum_{n=1}^{+\infty} (a_n \cos(nx) + b_n \sin(nx)) \cos(mx) \right.$

Oss: se $g_n \to g$ unif. in X e $h: X \to \mathbb{R}$ è limitata, allora: $g_n h \to gh$ unif. in X

$$|g_n h - gh| = |g_n - g| \underbrace{|h|}_{\leq M} \leq M |g_n - g|$$

$$\Rightarrow \underbrace{\sup_x |g_n h - g h|}_{\to 0} \leq M \cdot \underbrace{\sup_x |g_n - g|}_{\to 0} \quad \Leftarrow$$

Per l'osservazione : la serie in $\circledast$ converge uniformemente, quindi si può integrare t.a t.:

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = \underbrace{\int_{-\pi}^{\pi} a_0 \cos(mx)\, dx}_{=0} +$$

$$+ \sum_{n=1}^{+\infty} \int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx)) \cos(mx)\, dx$$

$$= \sum_{n=1}^{+\infty} \Big(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx}_{\substack{0 \quad n \neq m \\ \pi \quad n = m}} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx}_{=0 \ \forall n \geq 1} \Big)$$

$$= a_m \pi$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx = a_m \pi \quad \Rightarrow \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx$$

Moltiplicando per $\sin(mx)$ ottengo

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx \qquad \square$$

f 2π- periodica

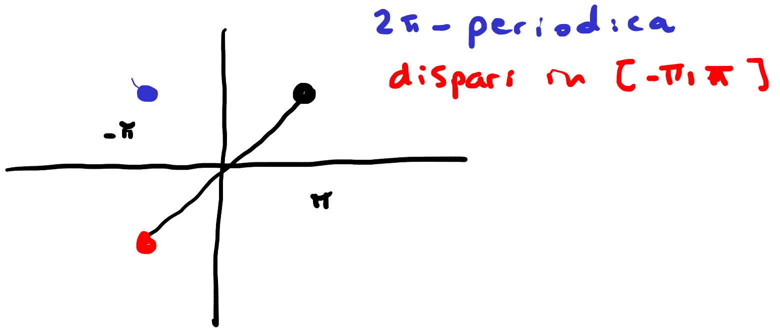
Posso dire: " f dispari in $[-\pi, \pi]$ " ?

O meglio: esistono funzioni 2π-periodiche t.c. $f|_{[-\pi,\pi]}$ è dispari ?

$$\left\{ \begin{array}{ll} f \text{ dispari} \Rightarrow & f(-\pi) = -f(\pi) \\ f \ 2\pi\text{-periodica} \Rightarrow & f(-\pi) = f(-\pi+2\pi) = f(\pi) \end{array} \right\}$$

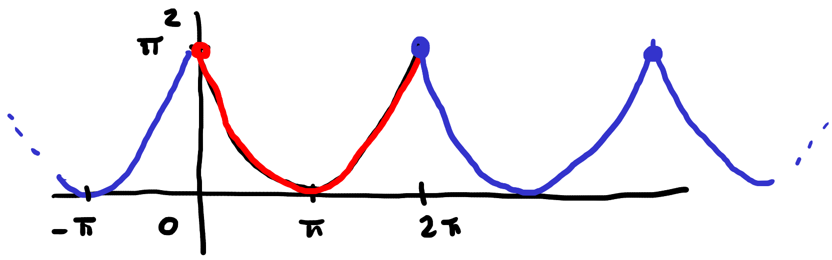
$$\Rightarrow -f(\pi) = f(\pi) \Rightarrow f(\pi) = 0.$$

È corretto chiedere: "$f|_{]-\pi,\pi[}$ dispari"



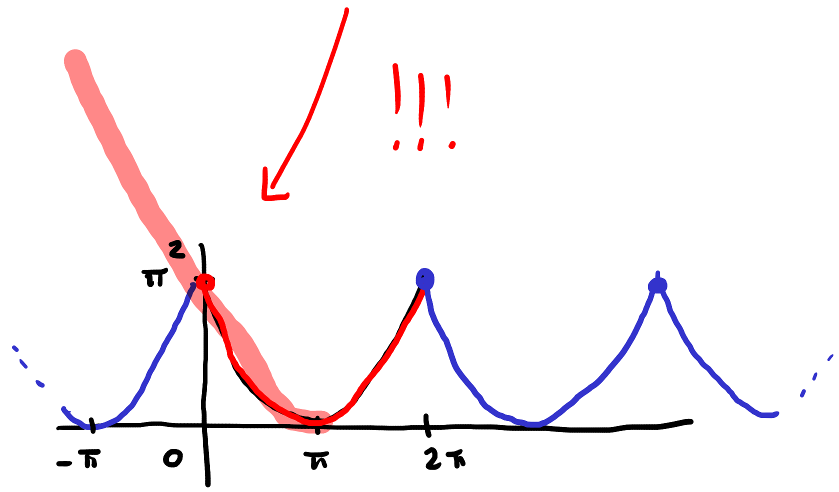
f: prolungamento 2π-periodico di

$x \in [0, 2\pi[\ \longmapsto (x-\pi)^2$



f continua in $\mathbb{R}$ $\Rightarrow$ i coeff. di Fourier sono ben definiti

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} (x-\pi)^2\,dx}_{} = \dots$$



Procedimento corretto:

- integro in $[0, 2\pi]$ (invece che in $[-\pi, \pi]$)

  oppure:

- osservo che f è pari, quindi:

$$b_n = 0 \quad \forall n \geq 1$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(x-\pi)^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(x-\pi)^3}{3}\right]_0^{\pi} = -\frac{1}{\pi}\left(-\frac{\pi^3}{3}\right) = \frac{\pi^2}{3}$$

$\forall n \geq 1$:

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x-\pi)^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\underbrace{\left[(x-\pi)^2\frac{\sin(nx)}{n}\right]_0^{\pi}}_{=0} - \int_0^{\pi} 2(x-\pi)\frac{\sin(nx)}{n}\,dx\right)$$

$$= -\frac{4}{\pi n}\int_0^{\pi}(x-\pi)\sin(nx)\,dx$$

$$= -\frac{4}{\pi n}\left(\left[(x-\pi)\left(-\frac{\cos(nx)}{n}\right)\right]_0^{\pi} - \underbrace{\int_0^{\pi}\left(-\frac{\cos(nx)}{n}\right)dx}_{=0}\right)$$

$$= -\frac{4}{\pi n}\left(+\pi\left(-\frac{1}{n}\right)\right) = +\frac{4}{n^2}$$

$\Rightarrow$ La serie di Fourier di f è

$$\frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2}\cos(nx)$$

Converge? Sì, totalmente in $\mathbb{R}$

$\sum_n \frac{4}{n^2}$ conv. (assol.)

Definisco $F(x) := \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2}\cos(nx)$

Domanda: $F(x) = f(x)$?

In questo momento: non so rispondere

Se sapessi che la risposta è affermativa, avrei

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2}\cos(nx) \quad \forall x \in \mathbb{R}$$

In particolare, per $x = 0$:

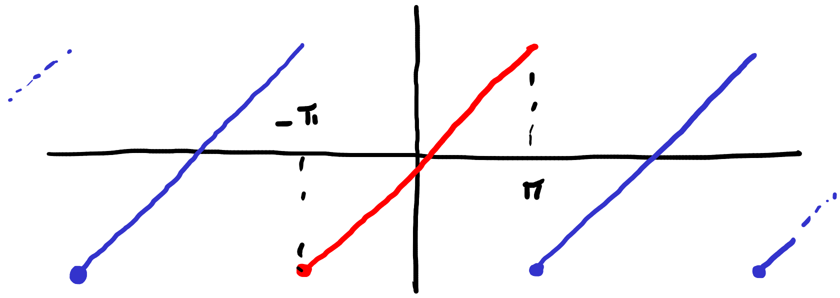
$$\underset{\pi^2}{\overset{}{f(0)}} = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \cdot \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{2}{3}\pi^2 \cdot \frac{1}{4} = \frac{\pi^2}{6}$$

f : prolungamento 2π - periodico di:

$$x \in [-\pi, \pi[\longmapsto x$$



$f|_{]-\pi,\pi[}$ è dispari $\Rightarrow a_n = 0 \quad \forall n \in \mathbb{N}$

$\forall n \geq 1$: $\quad b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx$

$$= \frac{2}{\pi} \left(\left[x \left(-\frac{\cos(nx)}{n} \right) \right]_0^{\pi} - \underbrace{\int_0^{\pi} \left(-\frac{\cos(nx)}{n} \right) dx}_{=0} \right)$$

$$= \frac{2}{\pi} \frac{\pi(-\cos(n\pi))}{n} = -\frac{2}{n} \cos(n\pi)$$

$$= -\frac{2}{n} (-1)^n = (-1)^{n-1} \frac{2}{n}$$

S. di Fourier: $\quad \sum_n \underbrace{(-1)^{n-1} \frac{2}{n}} \sin(nx)$

$\uparrow$ converge? Boh!

(non è soddisfatta la cond. suff.)