26 ottobre 2023

Calcolo un valore approssimato di: $\ln(0.7)$

Punto di partenza:

$$\forall x \in]-1,1] : \ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(0.7) = \ln(1+(-0.3)) = \ln\Big(1+\underbrace{\Big(-\frac{3}{10}\Big)}_{\in]-1,1]}\Big) =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} (-1)^n \left(\frac{3}{10}\right)^n \cdot \frac{1}{n}$$

$$= -\underbrace{\boxed{\sum_{n=1}^{+\infty} \frac{3^n}{n\,10^n}}}_{=:\,S} \quad ?$$

Non posso usare la stima del resto del criterio di Leibniz!

Osservo che

$$\forall n: \quad \frac{3^n}{n\,10^n} \leq \left(\frac{3}{10}\right)^n$$

$$\Rightarrow \quad \forall n: \quad R_n \leq R'_n \leftarrow$$

resto n-esimo della serie di termine $\left(\frac{3}{10}\right)^n$

$$\forall n: \quad R_n \leq \frac{\left(\frac{3}{10}\right)^{n+1}}{1-\frac{3}{10}}$$

$$\forall n: \quad R_n \leq \frac{10}{7}\cdot\left(\frac{3}{10}\right)^{n+1} = \frac{3}{7}\left(\frac{3}{10}\right)^n$$

Se voglio calcolare S con un errore minore di 10^{-3}, mi basta determinare n t.c.

$$\frac{3}{7}\cdot\left(\frac{3}{10}\right)^n < 10^{-3} \qquad (*)$$

10 $\iff \frac{7}{3}\left(\frac{10}{3}\right)^n > 10^3$

$\iff 7 \cdot 10^{n-3} > 3^{n+1}$

$n = 3: \quad 7 > 3^4$ no!

$n = 4: \quad 70 > 3^5$ no!

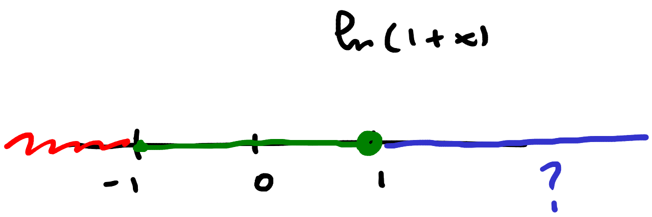
$n = 5: \quad 700 > 3^6 = 81 \cdot 9$ no!

$n = 6: \quad 7000 > 81 \cdot 9 \cdot 3$ sì!

$\Rightarrow S \simeq S_6 = \sum_{n=1}^{6} \frac{3^n}{n \cdot 10^n} = \dots = \boxed{\quad}$

$\ln(0.7) \simeq - \boxed{\quad}$

approssimazione con errore $< 10^{-3}$.



$\ln(2.5) = ?$

$x > 1 \Rightarrow 0 < \frac{1}{x} < 1$

$\ln\left(\frac{1}{x}\right) = \ln(1) - \ln(x) = -\ln(x)$

$x > 1 \Rightarrow 0 < \frac{1}{x} < 1 \Rightarrow$ calcolo $\ln\left(\frac{1}{x}\right) \dots$

$\Rightarrow \ln(x) = -\ln\left(\frac{1}{x}\right) \dots$

$\ln(2.5) = \ln\left(\frac{5}{2}\right) = \underbrace{-\ln\left(\frac{2}{5}\right)}$ come prima...

Calcolo $\int_0^1 e^{-x^2}\,dx$ con errore $< 10^{-3}$.

Punto di partenza:

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad e^{-x^2} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{n!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad \int_0^x e^{-t^2}\,dt \underset{\substack{\uparrow \\ \text{integro} \\ \text{termine} \\ \text{a termine}}}{=} \sum_{n=0}^{+\infty} \int_0^x (-1)^n \frac{t^{2n}}{n!}\,dt$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!} \left[\frac{t^{2n+1}}{2n+1} \right]_0^x$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad \int_0^x e^{-t^2}\,dt = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$$

Per $x = 1$:

$$\int_0^1 e^{-t^2}\,dt = \sum_{n=0}^{+\infty} (-1)^n \underbrace{\frac{1}{n!\,(2n+1)}}_{b_n}$$

$b_n > 0$

$b_n \to 0$

$b_n \downarrow$

cr. Leibniz

$$\Rightarrow \quad \forall n: \ |S - S_n| \le b_{n+1} = \frac{1}{(n+1)!\,(2n+3)}$$

Per determinare n t.c. $|S - S_n| < 10^{-3}$ mi basta determinare n t.c.

$$\frac{1}{(n+1)!\,(2n+3)} < 10^{-3}$$

cioè :

$$(n+1)!\,(2n+3) > 1000$$

$n = 3$ $\quad 4!\,9$ $\quad$ no

$\cancel{n = 5}$ $\quad 6!\,13 = 720 \cdot 13$ $\quad$ si

$\underline{n = 4}$? $\quad 5!\,11 = 120 \cdot 11$ $\quad$ si

$$\Rightarrow \int_0^1 e^{-x^2}\,dx \cong S_4 = 1 - \frac{1}{3} + \frac{1}{2!\,5} - \frac{1}{3!\,7} + \frac{1}{4!\,9}$$

$$= \dots \qquad \square$$

Calcolo $\int_0^1 \frac{\sin(x)}{x}\,dx$ con errore $< 10^{-4}$.

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Posso scriverla come somma di serie di potenze ?

Punto di partenza:

$$\forall x \in \mathbb{R} : \quad \sin(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

reg. multiplo

$$\Downarrow$$

$$\Rightarrow \forall x \in \mathbb{R}^* : \quad \frac{\sin(x)}{x} = \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}}_{=:\, g(x)} \quad , \ g(0) = 1$$

$g : \mathbb{R} \to \mathbb{R}$ continua

$\Rightarrow \quad \forall x \in \mathbb{R}^* : \quad f(x) \overset{\circledast}{=} g(x)$

sia f che g sono continue in $x = 0$

$\Rightarrow$ l'uguaglianza $\circledast$ vale anche in $x=0$

Conclusione:

$$\forall x \in \mathbb{R} : \quad f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$\Rightarrow \quad \forall x \in \mathbb{R}:$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} \int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!}\,dt$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!\,(2n+1)}$$

Per $x = 1$:

$$\int_0^1 f(t)\,dt = \sum_{n=0}^{+\infty} (-1)^n \underbrace{\frac{1}{(2n+1)!\,(2n+1)}}_{b_n} \quad \dots$$

Impongo $b_{n+1} < 10^{-4}$, cioè

$$(2n+3)!\,(2n+3) > 10^4 ;$$

vero per $n = 2 \quad \Rightarrow$

$$\int_0^1 \frac{\sin(x)}{x}\,dx \simeq S_2 = \sum_{n=0}^{2} (-1)^n \frac{1}{(2n+1)!\,(2n+1)}$$

$$= 1 - \frac{1}{3!\,3} + \frac{1}{5!\,5} = \dots = \frac{1703}{1800} = \underline{0.9461}1\dots$$

Wolfram Alpha: $\underline{0.946}083$

Calcolo $\int_0^1 \frac{1-\cos(x^2)}{x^3}\,dx$ con errore $< 10^{-4}$

Definisco
$$f(x) = \begin{cases} \frac{1-\cos(x^2)}{x^3} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

continua

$x \to 0$: $\frac{1-\cos(x^2)}{x^3} \sim \frac{1-\left(1-\frac{x^2}{2}+\ldots\right)}{x^3} \sim \frac{x}{2}+\cdots \xrightarrow[x\to 0]{} 0$

Verifico se f è somma di una serie di potenze.

Punto di partenza:

$$\forall x \in \mathbb{R}: \quad \cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow$$

$$\forall x \in \mathbb{R}: \quad \cos(x^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{4n}}{(2n)!} = 1 + \sum_{n=1}^{+\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad 1-\cos(x^2) = -\sum_{n=1}^{+\infty} (-1)^n \frac{x^{4n}}{(2n)!} = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{4n}}{(2n)!}$$

$$\Rightarrow \forall x \in \mathbb{R}^*: \quad \underbrace{\frac{1-\cos(x^2)}{x^3}}_{f(x)} = \underbrace{\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{4n-3}}{(2n)!}}_{=:\,g(x)}$$

$f(x)$: continua in $\mathbb{R}$ ($\Rightarrow$ in $x=0$)

$g(x)$: continua in $\mathbb{R}$ ($\Rightarrow$ in $x=0$)

Per continuità:

$$\forall x \in \mathbb{R}: \quad f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{4n-3}}{(2n)!}$$

Quindi:

$$\int_0^1 f(x)\,dx = \sum_{n=1}^{+\infty} (-1)^{n-1} \int_0^1 \frac{x^{4n-3}}{(2n)!}\,dx$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} \left[\frac{x^{4n-2}}{(2n)!\,(4n-2)} \right]_0^1$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{(2n)!\,(4n-2)} \quad b_n \quad \dots$$

Risolvo in serie l'eq. diff. $y'' + y = 0$ (soluzione già nota da AM2)

$a_1(t) \equiv 0$
$a_0(t) \equiv 1$ } analitiche in $t_0 = 0$, con rdc $= +\infty$

$\Rightarrow$ mi aspetto di trovare due sol. lin. indip. analitiche in $t_0=0$ con rdc $= +\infty$.

Scrivo la generica soluzione nella forma

$$y(t) = \sum_{n=0}^{+\infty} c_n t^n \qquad \forall t \in \mathbb{R}$$

$c_0 + c_1 t + c_2 t^2 + \dots$

$c_0 = y(0)$, $c_1 = y'(0)$

con c_n da determinare.

Derivo t.a t. due volte:

$$\forall t \in \mathbb{R}: \quad y'(t) = \sum_{n=1}^{+\infty} n c_n t^{n-1}, \quad y''(t) = \sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2}$$

Sostituisco nell'equazione:

$$\forall t \in \mathbb{R}: \quad \sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2} + \sum_{n=0}^{+\infty} c_n t^n = 0$$

Faccio delle manipolazioni algebriche; dato che le serie coinvolte convergono assolutamente, tutti i passaggi saranno leciti.

$$\forall t \in \mathbb{R}: \quad \sum_{n=0}^{+\infty} (n+2)(n+1)\, c_{n+2}\, t^n + \sum_{n=0}^{+\infty} c_n t^n = 0 \quad \Leftrightarrow$$

$$\forall t \in \mathbb{R}: \quad \sum_{n=0}^{+\infty} \big((n+2)(n+1) c_{n+2} + c_n\big)\, t^n = 0 = \sum_{n=0}^{+\infty} 0 \cdot t^n$$

Pr. Id. Serie Pot. $\Rightarrow$

$$\forall n: \quad (n+2)(n+1)\, c_{n+2} + c_n = 0$$

$$\Rightarrow \quad \forall n \geq 0 \quad c_{n+2} = -\frac{c_n}{(n+2)(n+1)} \qquad \odot$$

c_0 arbitrario $\quad (c_0 = y(0))$

c_1 arbitrario $\quad (c_1 = y'(0))$

$$c_2 = -\frac{c_0}{2 \cdot 1} \qquad\qquad c_3 = -\frac{c_1}{3 \cdot 2}$$

$$c_4 = -\frac{c_2}{4 \cdot 3} = +\frac{c_0}{4 \cdot 3 \cdot 2 \cdot 1} \qquad\qquad c_5 = -\frac{c_3}{5 \cdot 4} = +\frac{c_1}{5 \cdot 4 \cdot 3 \cdot 2}$$

$$c_6 = -\frac{c_4}{6 \cdot 5} = -\frac{c_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \qquad\qquad c_7 = -\frac{c_5}{7 \cdot 6} = -\frac{c_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}$$

$\vdots \qquad\qquad \vdots$

$$c_{2n} = (-1)^n \frac{c_0}{(2n)!} \qquad\qquad c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}$$

$$y(t) = \sum_{n=0}^{+\infty} c_n t^n =$$

$$= c_0 + c_1 t + \sum_{n=1}^{+\infty} (-1)^n \frac{c_0}{(2n)!} t^{2n} + \sum_{n=1}^{+\infty} (-1)^n \frac{c_1}{(2n+1)!} t^{2n+1}$$

$$= c_0\left(1+\sum_{n=1}^{+\infty}(-1)^n\frac{t^{2n}}{(2n)!}\right)+c_1\left(t+\sum_{n=1}^{+\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}\right)$$

$$= c_0\underbrace{\sum_{n=0}^{+\infty}(-1)^n\frac{t^{2n}}{(2n)!}}_{\cos(t)}+c_1\underbrace{\sum_{n=0}^{+\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)}$$

$\square$

Equazione di Airy

$$(*)\quad y''-ty=0 \qquad \left.\begin{array}{l} a_1(t)\equiv 0\\ a_0(t)=-t\end{array}\right\}\ \begin{array}{l}\text{analitiche}\\ \text{in } t=0\\ \text{rdc } =+\infty\end{array}$$

$$y(t)=\sum_{n=0}^{+\infty}c_nt^n \qquad \forall t\in\mathbb{R}$$

Derivo t.a t. due volte e sostituisco in $(*)$:

$$\forall t\in\mathbb{R}:\quad \sum_{n=2}^{+\infty}n(n-1)c_nt^{n-2}-t\sum_{n=0}^{+\infty}c_nt^n=0$$

$$\sum_{n=0}^{+\infty}(n+2)(n+1)c_{n+2}t^n-\sum_{n=0}^{+\infty}c_nt^{n+1}=0$$

$$\sum_{n=0}^{+\infty}(n+2)(n+1)c_{n+2}t^n-\sum_{n=1}^{+\infty}c_{n-1}t^n=0$$

$$2c_2+\sum_{n=1}^{+\infty}(n+2)(n+1)c_{n+2}t^n-\sum_{n=1}^{+\infty}c_{n-1}t^n=0$$

$$\forall t\in\mathbb{R}:\quad 2c_2+\sum_{n=1}^{+\infty}\left(\underline{(n+2)(n+1)c_{n+2}-c_{n-1}}\right)t^n=0=0+\sum_{n=1}^{+\infty}\underline{0}\cdot t^n$$

PISP $\Rightarrow$

$$2c_2=0$$

$$\forall n\geq 1:\quad (n+2)(n+1)c_{n+2}-c_{n-1}=0$$

$$c_{n+2}=\frac{c_{n-1}}{(n+1)(n+2)}$$

c_0 arbitrario, c_1 arbitrario, $c_2 = 0$

$\Downarrow$

$c_5 = 0$

$\Downarrow$

$c_8 = 0$

$\vdots$

$c_{3n+2} = 0 \quad \forall n$

$$c_3 = \frac{c_0}{2 \cdot 3}$$

$$c_6 = \frac{c_3}{5 \cdot 6} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6}$$

$$c_9 = \frac{c_6}{8 \cdot 9} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$$

$\vdots$

$n \geq 1$:

$$c_{3n} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot (3n-1)(3n)}$$

$$\forall n \geq 1: \qquad c_{n+2} = \frac{c_{n-1}}{(n+1)(n+2)}$$

c_1 arbitrario

$$c_4 = \frac{c_1}{3 \cdot 4} \qquad c_7 = \frac{c_4}{6 \cdot 7} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}$$

$$c_{10} = \frac{c_7}{9 \cdot 10} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \; 9 \cdot 10}$$

$$\forall n \geq 1: \qquad c_{3n+1} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \ldots \cdot (3n)(3n+1)}$$

Quindi, $\forall t \in \mathbb{R}$:

$$y(t) = c_0 + c_1 t + \underbrace{c_2}_{=0} t^2 + \sum_{n=1}^{+\infty} c_{3n} t^{3n} + \sum_{n=1}^{+\infty} c_{3n+1} t^{3n+1} + \sum_{n=1}^{+\infty} \underbrace{c_{3n+2}}_{=0} t^{3n+2}$$

$$= \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot (3n-1)(3n)}$$

$$= c_0 \Big(\underbrace{1 + \sum_{n=1}^{+\infty} \frac{1}{2\cdot 3\cdot 5\cdot 6 \cdot \ldots (3n-1)(3n)} t^{3n}}_{=: y_1(t) \quad (y_1(0)=1,\ y_1'(0)=0)} \Big) +$$

$$+ c_1 \Big(\underbrace{t + \sum_{n=1}^{+\infty} \frac{1}{3\cdot 4\cdot 6\cdot 7 \cdot \ldots \cdot (3n)(3n+1)} t^{3n+1}}_{y_2(t) \quad (y_2(0)=0,\ y_2'(0)=1)} \Big)$$

$$(\underbrace{t^2+1}_{\neq 0\ \forall t \in \mathbb{R}})\, y'' + t y' - y = 0$$

$$y'' + \frac{t}{t^2+1} y' - \frac{1}{t^2+1} y = 0$$

$$a_0(t) = -\frac{1}{t^2+1} \underset{\substack{\uparrow \\ \text{lezione scorsa}}}{=} - \sum_{n=0}^{+\infty} (-1)^n t^{2n} = \sum_{n=0}^{+\infty} (-1)^{n+1} t^{2n}$$

$\Rightarrow a_0$ è analitica in $t=0$ rdc 1

$$a_1(t) = \frac{t}{t^2+1} = t \sum_{n=0}^{+\infty} (-1)^n t^{2n} = \sum_{n=0}^{+\infty} (-1)^n t^{2n+1}$$

$\Rightarrow a_1$ è analitica in $t=0$ rdc $=1$

$\Rightarrow$ l'eq. ha due sol. linearmente indipendenti analitiche in $t_0=0$ con rdc ≥ 1

$$\forall t \in]-1,1[: \quad y(t) = \sum_{n=0}^{+\infty} c_n t^n$$

Derivo t. a t. e sostituisco nell'equazione:

$$\underbrace{\sum_{n=2}^{+\infty} n(n-1) c_n t^{n-2}}_{y''(t)} + \Big(\underbrace{\sum_{n=0}^{+\infty} (-1)^n t^{2n+1}}_{a_1(t)}\Big)\Big(\underbrace{\sum_{n=1}^{+\infty} n c_n t^{n-1}}_{y'(t)}\Big) +$$

$$+\underbrace{\left(\sum_{n=0}^{+\infty}(-1)^{n+1}t^{2n}\right)}_{a_0(t)}\underbrace{\left(\sum_{n=0}^{+\infty}c_n t^n\right)}_{y(t)} = 0$$

Come si moltiplicano due serie di potenze?
Più in generale: come si moltiplicano tra loro due serie numeriche?

Digressione sulle serie numeriche

$$\sum_{n=0}^{+\infty} a_n =: \underset{\in\mathbb{R}}{A}, \qquad \sum_{n=0}^{+\infty} b_n =: \underset{\in\mathbb{R}}{B}$$

Serie "somma": $\displaystyle\sum_{n=0}^{+\infty}(a_n+b_n) = A+B$ (noto)

Serie "prodotto"? $\displaystyle\underbrace{\sum_{n=0}^{+\infty}(a_n\cdot b_n)}_{\text{converge?}} \overset{??}{=} A\cdot B \longrightarrow$ No!

Esempio:

$$\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

$$\sum_{n=0}^{+\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

$$2\cdot\frac{3}{2} = 3$$

$$\neq$$

$$\sum_{n=0}^{+\infty}\left(\left(\frac{1}{2}\right)^n\cdot\left(\frac{1}{3}\right)^n\right) = \sum_{n=0}^{+\infty}\left(\frac{1}{6}\right)^n = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$$

Se voglio che la somma della "serie prodotto" sia uguale al prodotto delle somme delle due serie, devo cambiare definizione.

Date $\{a_n\}$ e $\{c_n\}$, definisco $c_n := \sum_{k=0}^{n} a_k b_{n-k}$

$$c_0 = a_0 b_0 \,, \quad c_1 = \sum_{k=0}^{1} a_k b_{1-k} = a_0 b_1 + a_1 b_0$$

$$c_2 = \sum_{k=0}^{2} a_k b_{2-k} = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$(a_0 + a_1 + a_2 + \dots)(b_0 + b_1 + b_2 + \dots)$$

La serie di termine c_n si chiama serie **prodotto secondo Cauchy** delle serie di termini a_n e b_n.

Da completare ...