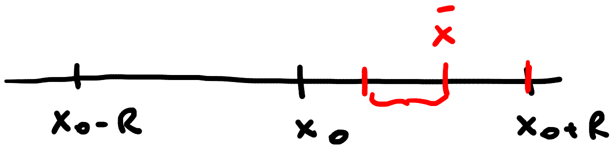
$$f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n \qquad x \in \,]x_0 - R, x_0 + R[$$



Oss: una funzione polinomiale può sempre essere "ricentrata"

Es: $p(x) = \underbrace{2}_{c_2} x^2 + \underbrace{5}_{c_1} x + \underbrace{1}_{c_0} \qquad x_0 = 0$

Prendo $\bar{x} = 2$:

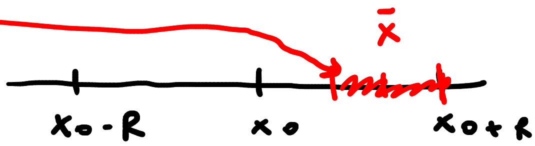
$(x-2)^2 + 4(x-2) + 4$

$$p(x) = 2(x-2+2)^2 + 5(x-2+2) + 1$$

$$= 2(x-2)^2 + 8(x-2) + 8 + 5(x-2) + 10 + 1$$

$$= \underbrace{2}_{\tilde{c}_2}(x-2)^2 + \underbrace{13}_{\tilde{c}_1}(x-2) + \underbrace{19}_{\tilde{c}_0} \qquad \bar{x} = 2$$

$$f(x) = \sum_{n=0}^{+\infty} c_n (x-x_0)^n$$



$$= \sum_{n=0}^{+\infty} c_n (x - \bar{x} + \bar{x} - x_0)^n$$

$$= \sum_{n=0}^{+\infty} c_n \sum_{k=0}^{n} \binom{n}{k} \underbrace{(x-\bar{x})^k}_{\text{non dip. da } n} (\bar{x} - x_0)^{n-k}$$

$$\overset{?}{=} \sum_{k=0}^{+\infty} \underbrace{\left(\sum_{n=k}^{+\infty} c_n \binom{n}{k} (\bar{x}-x_0)^{n-k} \right)}_{=: \tilde{c}_k} (x-\bar{x})^k$$

sì, perché la serie a due indici converge <u>assolutamente</u>

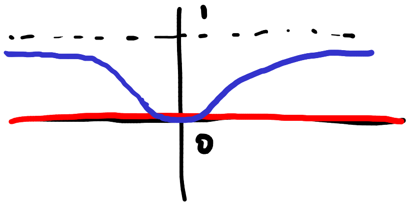
Es. di funzione di classe C^∞ <u>non</u> analitica:

$$f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$$

$$\forall n: \quad \exists f^{(n)}(0) = 0 \quad \Rightarrow \quad \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n \equiv 0 \neq f(x)$$

$\uparrow$

$\forall x \neq 0.$



Applico la cond. suff. a sin, cos, exp.

- $\forall n, \forall x \in \mathbb{R}: \quad |\sin^{(n)}(x)| \leq 1$ ✓
- $\forall n: \forall x \in \mathbb{R}: \quad |\cos^{(n)}(x)| \leq 1$ ✓
- $f(x) = e^x \qquad \forall n: \quad f^{(n)}(x) = e^x$

Fisso $\delta > 0$:

$$\forall x \in [-\delta, \delta], \ \forall n: \qquad |f^{(n)}(x)| \leq e^{\delta} =: M$$

cond. suff.

$$\Rightarrow \qquad \forall x \in [-\delta, \delta]: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

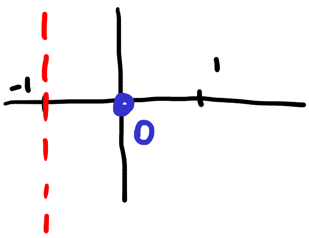
$\leftarrow$ nota da AM I

δ arbitrario $\Rightarrow$

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad \square$$

- $f(x) = \ln(1+x)$

$x \in]-1, 1[$



$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$$

$f''(x) = -(1+x)^{-2}$ $\qquad$ $f'''(x) = 2(1+x)^{-3}$ $\quad \dots$

Le derivate non sono limitate in $]-1,1[$

$\Rightarrow$ la successione delle derivate non è equilimitata in $]-1,1[$

$\Rightarrow$ la cond. suff. non è applicabile.

Dimostro la cond. suff. per l'analiticità.

Ipotesi: $\exists\, M, \delta \in \mathbb{R}_+^*$ t.c.

$$\forall k \in \mathbb{N},\ \forall x \in]x_0-\delta, x_0+\delta[:\quad |f^{(k)}(x)| \le M.$$

Tesi: $\forall x \in]x_0-\delta, x_0+\delta[:\quad \displaystyle\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x)$

$$\forall n: \quad T_{n,x_0}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

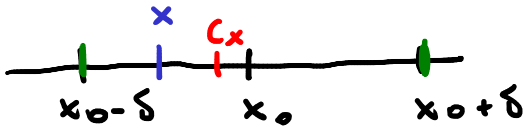
Tesi $\Leftrightarrow \forall x \in]x_0-\delta, x_0+\delta[:\quad \lim_{n\to+\infty} T_{n,x_0}(x) = f(x)$

$\Leftrightarrow \qquad$ " $\qquad : \quad \lim_{n\to+\infty} |T_{n,x_0}(x) - f(x)| = 0$

Fisso $x \in]x_0-\delta, x_0+\delta[$.
Ricordo, dall'Analisi I, che $\exists\, c_x$ compreso tra x e x_0 t.c.

$$f(x) - T_{n,x_0}(x) =: R_{n,x_0}(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

$$\Rightarrow \quad |f(x) - T_{n,x_0}(x)| = \frac{|f^{(n+1)}(c_x)|}{(n+1)!}|x-x_0|^{n+1} \underset{\text{ipotesi}}{\le} \frac{M}{(n+1)!}|x-x_0|^{n+1}$$



Dunque:

$$\forall x \in \left]x_0-\delta, x_0+\delta\right[: \quad 0 \leq |T_{n,x_0}(x) - f(x)| \leq M \frac{|x-x_0|^{n+1}}{(n+1)!}$$

Ricordo che per ogni $a>0$:

$$\lim_{n\to+\infty} \frac{a^n}{n!} = 0$$

$\xrightarrow{\text{TCO}}$

$$\Rightarrow \quad \forall x \in \left]x_0-\delta, x_0+\delta\right[: \quad \lim_{n\to+\infty} |T_{n,x_0}(x) - f(x)| = 0$$

$\square$

Verifico che $f(x) = \ln(1+x)$ è analitica in $x_0 = 0$.

Parto con l'osservare che

$$\forall x \in \left]-1,1\right[: \quad \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$$

$$\Rightarrow \quad \forall x \in \left]-1,1\right[: \quad \frac{1}{1+x} = \frac{1}{1-\underbrace{(-x)}_{\in\,]-1,1[}} = \sum_{n=0}^{+\infty} (-x)^n$$

$$\Rightarrow \quad \forall x \in \left]-1,1\right[: \quad \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$$

integro t.a t

$$\Rightarrow \quad \forall x \in \left]-1,1\right[: \quad \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{+\infty} \int_0^x (-1)^n t^n\,dt$$

$$\Rightarrow \quad \forall x \in \left]-1,1\right[: \quad \left[\ln(1+t)\right]_0^x = \sum_{n=0}^{+\infty} (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \quad \forall x \in \left]-1,1\right[: \quad \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$\Rightarrow \quad \forall x \in]-1,1[\; : \quad \underbrace{\ln(1+x)}_{f(x)} = \underbrace{\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}}_{g(x)}$

Oss:

- per $x = -1$: f non è definita e la serie a secondo membro diventa $\sum_n (-1)^{n-1} \frac{(-1)^n}{n} = \sum_n \left(-\frac{1}{n}\right)$ non converge

- per $x = 1$: $f(1) = \ln(2)$

  $\sum_n (-1)^{n-1} \frac{1}{n}$ converge (per Leibniz)

  $\Rightarrow g$ è continua in $]-1,1]$

Ovviamente: anche f è continua in $]-1,1]$

Ricapitolando:

$$\underbrace{\forall x \in]-1,1[\; : \quad f(x) = g(x)}$$

$$f(1) \underset{\substack{\uparrow \\ f \text{ continua} \\ \text{in } 1}}{=} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} g(x) \underset{\substack{\uparrow \\ g \text{ continua} \\ \text{in } 1}}{=} g(1)$$

Quindi:

$$\log(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n} \quad \forall x \in]-1,1]$$

□

Verifico che $f(x) = \arctan(x)$ è analitica in $x_0 = 0$.

Riprendo l'uguaglianza:

$$\forall x \in]-1,1[: \quad \frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$$

$$\Rightarrow \forall x \in]-1,1[: \quad \frac{1}{1+\underbrace{x^2}_{\in [0,1[\subset]-1,1[}} = \sum_{n=0}^{+\infty} (-1)^n (x^2)^n = \sum_{n=0}^{+\infty} (-1)^n x^{2n}$$

$$\Rightarrow \forall x \in]-1,1[: \quad \int_0^x \frac{1}{1+t^2}\, dt \underset{\substack{\text{integr.} \\ \text{t. a t.}}}{=} \sum_{n=0}^{+\infty} \int_0^x (-1)^n t^{2n}\, dt$$

$$\Rightarrow \forall x \in]-1,1[: \quad \arctan(x) \underset{\circledast}{=} \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}_{\dots \quad R = 1}$$

Per $x = \pm 1$: • f è definita (f è definita in $\mathbb{R}$)

• la serie a 2° membro diventa

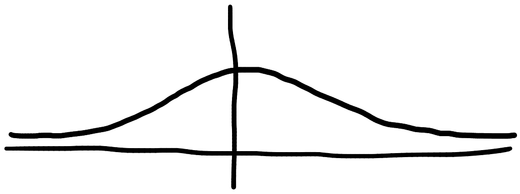
$$\sum_n (-1)^n \frac{(\pm 1)}{2n+1} = \pm \underbrace{\sum_n (-1)^n \cdot \frac{1}{2n+1}}_{\substack{\text{converge} \\ \text{(Leibniz)}}}$$

$\Rightarrow$ converge.

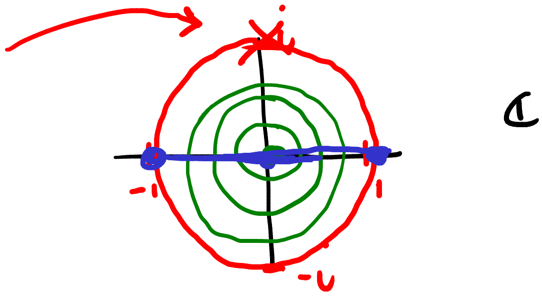
Con lo stesso ragionamento di prima, per continuità, l'uguaglianza $\circledast$ vale anche in $x = \pm 1$, quindi:

$$\forall x \in [-1,1] : \quad \arctan(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$x \mapsto \dfrac{1}{1+x^2}$



$z \in \mathbb{C} \mapsto \dfrac{1}{1+z^2}$



Verifico che sinh e cosh sono analitiche in $x_0 = 0$ con $R = +\infty$

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Punto di partenza:

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\Rightarrow \forall x \in \mathbb{R}: \quad e^{-x} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!}$$

serie convergenti (assolut.) in $\mathbb{R}$

regola della somma per serie convergenti $\Rightarrow$

$$\forall x \in \mathbb{R}: \quad e^x + e^{-x} = \sum_{n=0}^{+\infty} \left(\frac{x^n}{n!} + (-1)^n \frac{x^n}{n!} \right)$$

$$= \sum_{n=0}^{+\infty} \frac{1 + (-1)^n}{n!} x^n$$

regola multiplo ...

$$\Rightarrow \quad \forall x \in \mathbb{R}: \quad \cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{1+(-1)^n}{2\, n!} x^n$$

$$\frac{1+(-1)^n}{2} = \begin{cases} 1 & n \text{ pari} \\ 0 & n \text{ disp.} \end{cases}$$

$$= \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}$$

Analogamente:

$$= \begin{cases} 0 & n \text{ pari} \\ 1 & n \text{ disp.} \end{cases}$$

$$\forall x \in \mathbb{R}: \quad \sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{+\infty} \frac{1-(-1)^n}{2\, n!} x^n$$

$$= \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \square$$

Calcolo $\cos(0.5)$ con errore $< 10^{-3}$

Parto da:

$$\forall x \in \mathbb{R}: \quad \cos(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \quad \cos(0.5) = \cos\left(\frac{1}{2}\right) = \sum_{n=0}^{+\infty} (-1)^n \frac{\left(\frac{1}{2}\right)^{2n}}{(2n)!}$$

?

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{4^n (2n)!} =: b_n$$

Voglio determinare n t.c.

$$|S - S_n| < 10^{-3}$$

S: somma della serie

S_n: Somma parziale n-esima

Osservo che

- $b_n > 0 \quad \forall n$
- $b_n \to 0$
- $\{b_n\}$ decresc. (reciproco di succ crescente)

$\Rightarrow \sum_n (-1)^n b_n$ soddisfa le ipotesi del criterio

di Leibniz, quindi:

① la serie converge (lo sapevo già)

② $\forall n: \quad |S - S_n| \leq b_{n+1} = \dfrac{1}{4^{n+1}(2n+2)!}$

Perciò: se voglio che

$$|S - S_n| < 10^{-3}$$

mi basta determinare n in modo che

$$\frac{1}{4^{n+1}(2n+2)!} < 10^{-3}$$

cioè:

$$\underbrace{4^{n+1} \quad (2n+2)!} > 10^{3}$$

Provo: $n=1$: $4^2 \cdot 4! = 16 \cdot 24$ no!

$n=2$: $4^3 \cdot 6! = 64 \cdot 720 > 10^3$

$$\Rightarrow \quad S_2 = \sum_{k=0}^{2} (-1)^k \frac{1}{4^k (2k)!}$$

$$= 1 - \frac{1}{4 \cdot 2} + \frac{1}{16 \cdot 24} = \dots = \frac{337}{384} = 0.87760\dots$$

approssima S (= $\cos(0.5)$) con errore minore di 10^{-3}

Excel: $0.877582\dots$

Calcolo $\ln(1.2)$ con errore $< 10^{-3}$

Parto da $\forall x \in]-1,1]$: $\ln(1+x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$ ⊗

$$\ln(1.2) = \ln(1+0.2) = \ln\Big(1+\underbrace{\tfrac{1}{5}}_{\in\,]-1,1]}\Big)$$

$$\stackrel{(*)}{=} \sum_{n=1}^{+\infty} (-1)^{n-1} \underbrace{\frac{1}{5^n n}}_{b_n}$$

- $b_n > 0 \quad \forall n$
- $b_n \to 0$
- $\{b_n\}$ decresc.

cr. Leibniz $\Rightarrow$

$$\underbrace{|R_n|}_{S - S_n} \leq \frac{1}{5^{n+1}(n+1)}$$

Se voglio che $|S - S_n| < 10^{-3}$, mi basta imporre che

$$\frac{1}{5^{n+1}(n+1)} < 10^{-3}$$

cioè: $5^{n+1}(n+1) > 1000$

$n=1 \quad 5^2 \cdot 2 \quad$ no

$n=2 \quad 5^3 \cdot 3 \quad$ no

$n=3 \quad 5^4 \cdot 4 = 625 \cdot 4 \quad$ si!

$$\Rightarrow \ln(1.2) \simeq \sum_{k=1}^{3} (-1)^{k-1} \frac{1}{5^k \cdot k}$$

con errore $< 10^{-3}$

$$= \frac{1}{5} - \frac{1}{50} + \frac{1}{375} = \dots = \frac{137}{750}$$

$$= \underbrace{0.182}6\dots$$

Excel: $\underbrace{0.182}321157$