19 ottobre 2023

$$\sum_n \frac{x^{2n}}{n} \quad (1) \qquad x_0 = 0$$

$$c_0 = 0$$
$$c_n = \begin{cases} \frac{1}{n/2} & n \text{ pari} \\ 0 & n \text{ dispari} \end{cases}$$

$$c_n x^n$$

$$c_{2n} = \frac{1}{n}$$
$$c_{2n+1} = 0$$

$$= \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3} + \dots$$

$n \geq 1$

$$\sqrt[n]{|c_n|} = \begin{cases} 0 & n \text{ dispari} \\ \sqrt[n]{\frac{2}{n}} & n \text{ pari} \end{cases}$$

$$\text{"}\lim_{n\to+\infty}\text{"} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} \sqrt[n]{\frac{2}{n}} = \underbrace{1}_{=:\alpha}$$

$$\Rightarrow R = 1$$

$x = \pm 1$: $\sum_n \frac{1}{n}$ div.

$\Rightarrow$ c. punt/assol. $]-1,1[$ (nessuna conv. in $\mathbb{R} \setminus]-1,1[$)

c. unif/tot. $[a,b]$

$-1 < a < b < 1$

In alternativa: pongo $t = x^2$ e studio

(2) $\sum_n \frac{t^n}{n}$ $\qquad c_n = \frac{1}{n} \quad \forall n \geq 1$

$c_0 = 0$

$$\text{"}\lim\text{"} \sqrt[n]{|c_n|} = \dots = 1 \quad \Rightarrow R = 1$$

$t = 1$: $\sum_n \frac{1}{n}$ div.

$t = -1$: $\sum_n \frac{(-1)^n}{n}$ conv. (Leibniz)

(2) : conv. punt. $[-1,1[$ $\qquad$ unif. $[-1,b]$, $-1 < b < 1$

assol. $]-1,1[$ $\qquad$ tot. $[a,b]$

$-1 < a < b < 1$

①: conv. punt. in $\{x \in \mathbb{R} \mid x^2 \in [-1,1[\} =$

$$]-1,1[$$

conv. assol. in $\{x \in \mathbb{R} \mid x^2 \in]-1,1[\} =$

$$]-1,1[$$

conv. unif in $\{x \in \mathbb{R} \mid x^2 \in [-1,b],\ 0<b<1$

$$= [-\sqrt{b}, \sqrt{b}] \qquad b<1$$

$$= [-\beta, \beta] \qquad \beta<1$$

conv. tot in $\{x \in \mathbb{R} \mid x^2 \in [a,b],\ -1<\overset{<0<}{a}<b<1\}$

$$= [-\sqrt{b}, \sqrt{b}]$$

$$= [-\beta, \beta] \qquad \beta<1$$

① $\sum_n \frac{(-1)^n n^2}{n^4+2} \left(\frac{x^2-3}{x+2}\right)^n$ $\qquad t = \frac{x^2-3}{x+2}$ $\quad (x \in \mathbb{R} \setminus \{-2\})$

② $\sum_n \frac{(-1)^n n^2}{n^4+2} t^n$ $\qquad t_0 = 0,\quad c_n = (-1)^n \frac{n^2}{n^4+2}$

$$\lim_{n\to+\infty} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} \sqrt[n]{\frac{n^2}{n^4+2}} = \lim_{n\to+\infty} \frac{\sqrt[n]{n^2} \to 1}{\sqrt[n]{n^4+2} \to} = 1 =: \alpha$$

$\Rightarrow R = 1$

$t = 1 \qquad \sum_n (-1)^n \frac{n^2}{n^4+2}$

$t = -1 \qquad \sum_n (-1)^n \frac{n^2}{n^4+2} (-1)^n = \sum_n \frac{n^2}{n^4+2}$ converge (confr. asint)

$$\sim \frac{n^2}{n^4} = \frac{1}{n^2}$$

In $t=-1$ c'è conv., che equivale a conv. assol.

$\Rightarrow$ la serie (2) conv. totalm. in $[-1,1]$

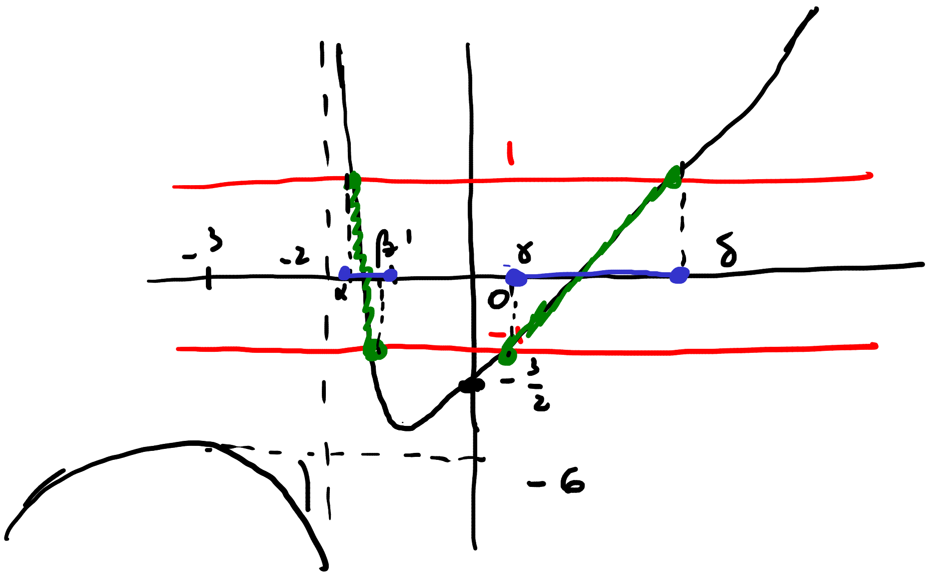
(e non converge in alcun modo al di fuori di questo intervallo).

Pertanto: la serie (1) converge totalmente in

$$\left\{x \in \mathbb{R} \setminus \{-2\} \;\middle|\; \frac{x^2-3}{x+2} \in [-1,1]\right\}$$

(e non converge altrove)

$$g(x) := \frac{x^2-3}{x+2}$$



$g(-2^-) = -\infty$

$g(-2^+) = +\infty$

$g(\pm\infty)$

$g(0) = -\frac{3}{2}$

$$g'(x) = \frac{2x(x+2)-(x^2-3)}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2} = 0 \qquad \begin{array}{l} x=-1 \\ x=-3 \end{array}$$

-3 punto di max loc

-1 " " min loc.

$g(-3) = -6$ $\qquad$ $g(-1) = -2$ $\leftarrow$ superfluo

La serie (1) conv. totalm. in $[\alpha, \beta] \cup [\gamma, \delta]$

Determino α e δ: $\quad g(x) = 1$

$$\frac{x^2-3}{x+2} = 1 \iff x^2-3 = x+2$$

$$x^2 - x - 5 = 0$$

$$x = \frac{1 \mp \sqrt{1+20}}{2}$$

$$\alpha = \frac{1-\sqrt{21}}{2} \qquad \delta = \frac{1+\sqrt{21}}{2}$$

Determino β e γ: $\quad g(x) = -1$

$$\frac{x^2-3}{x+2} = -1 \iff x^2 - 3 = -x-2$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \mp \sqrt{1+4}}{2}$$

$$\beta = \frac{-1-\sqrt{5}}{2} \qquad \gamma = -\frac{1+\sqrt{5}}{2}$$

Conv. tot. in $\left[\frac{1-\sqrt{21}}{2}, -\frac{(1+\sqrt{5})}{2}\right] \cup \left[\frac{\sqrt{5}-1}{2}, \frac{1+\sqrt{21}}{2}\right]$

① $\displaystyle\sum_n \frac{(x^2-3)^n e^{nx}}{e^{2n}+1}$ $\qquad t = (x^2-3)e^x$

② $\displaystyle\sum_n \frac{1}{e^{2n}+1} t^n$ $\qquad c_n = \frac{1}{e^{2n}+1} \quad \forall n$

$$\frac{|c_{n+1}|}{|c_n|} = \frac{1}{e^{2n+2}+1}(e^{2n}+1) \sim \frac{e^{2n}}{e^{2n+2}} = \frac{1}{e^2}$$

$$\Rightarrow R = e^2$$

$t = e^2$: $\quad \displaystyle\sum_n \frac{1}{e^{2n}+1} e^{2n} \to 1$ $\qquad$ non conv.

$t = -e^2$: $\quad \sum_n (-1)^n \left(\frac{e^{2n}}{e^{2n}+1}\right)$ $\to 1$ $\not\to 0$ non conv.

Per ② :

conv. punt/ass $\quad]-e^2, e^2[$

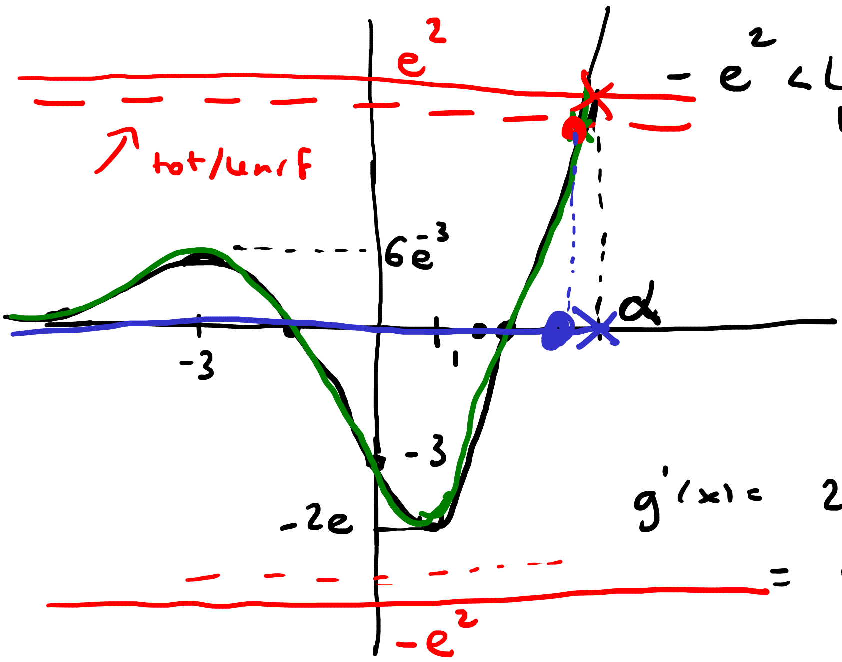
conv. unif/tot $\quad [a,b]$, $\quad -e^2 < a < b < e^2$

$-e^2 \quad e^2$

Per ① :

per quali x : $(x^2-3)e^x \in]-e^2, e^2[$??

$-e^2 < \underbrace{(x^2-3)e^x}_{=:g(x)} < e^2$??



$g(+\infty) = +\infty$

$g(-\infty) = 0$ (gerarchia infiniti)

$g'(x) = 2xe^x + (x^2-3)e^x$

$= e^x(x^2+2x-3) = 0$

$x = -3, \quad x = 1$

$g(-3) = 6e^{-3} < 1$

$g(1) = -2e$

Per ① : conv. punt/assol. in $]-\infty, \alpha[$

α è l'unica sol. di $g(x) = e^2$

$(x^2-3)e^x = e^2$

$\Rightarrow \alpha = 2$

Dal grafico:

conv. unif/tot in $]-\infty, a]$ $\forall a < 2$.

① $\sum_n \frac{n2^n+1}{n^2}\left(\frac{x}{x^2+1}\right)^n$ $\qquad t = \frac{x}{x^2+1}$

② $\sum_n \frac{n2^n+1}{n^2} t^n$ $\qquad C_n := \frac{n2^n+1}{n^2}$

$$\frac{|C_{n+1}|}{|C_n|} = \frac{(n+1)2^{n+1}+1}{(n+1)^2} \cdot \frac{n^2}{n2^n+1}$$

$$\sim \frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = \frac{2n}{n+1} \to 2$$

$\Rightarrow R = \frac{1}{2}$

$t = \frac{1}{2}$: $\quad \sum_n \frac{n2^n+1}{n^2} \cdot \left(\frac{1}{2}\right)^n = \sum_n \frac{n2^n+1}{n^2 2^n}$ div.

$\sim \frac{1}{n}$

$t = -\frac{1}{2}$: $\quad \sum_n (-1)^n \underbrace{\frac{n2^n+1}{n^2 2^n}}_{=:b_n}$ (*)

$b_n \sim \frac{1}{n} \Rightarrow b_n \to 0$ ✓

monotonia? $\quad b_n = \frac{n2^n}{n^2 2^n} + \frac{1}{n^2 2^n} = \underbrace{\frac{1}{n}}_{\text{decr.}} + \frac{1}{\underbrace{n^2}_{\text{cresc}}\underbrace{2^n}_{\text{cresc.}}}$ decresc.

$n^2 \uparrow$, $2^n \uparrow \Rightarrow n^2 2^n \uparrow$

$\Rightarrow \frac{1}{n^2 2^n} \downarrow$

$\Rightarrow \{b_n\}$ è decrescente

$\Rightarrow$ ⊛ converge (per Leibniz)

Per ②: conv. punt. $\left[-\frac{1}{2}, \frac{1}{2}\right[$

assol. $\left]-\frac{1}{2}, \frac{1}{2}\right[$

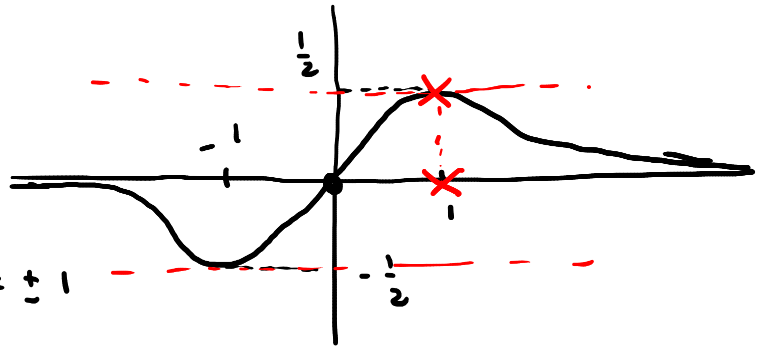
unif. $\left[-\frac{1}{2}, b\right]$ $\quad 0 < b < \frac{1}{2}$

tot. $[a, b]$ $\quad -\frac{1}{2} < a < 0 < b < \frac{1}{2}$

Per ①: $g(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right[$?? $\qquad g(x) = \frac{x}{x^2+1}$

$$g'(x) = \frac{x^2+1 - x \cdot 2x}{(x^2+1)^2}$$

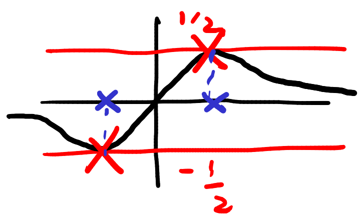
$$= \frac{1-x^2}{(x^2+1)^2} = 0 \iff x = \pm 1$$



$g(1) = \frac{1}{2}$

Conv. punt: $\mathbb{R} \setminus \{1\}$

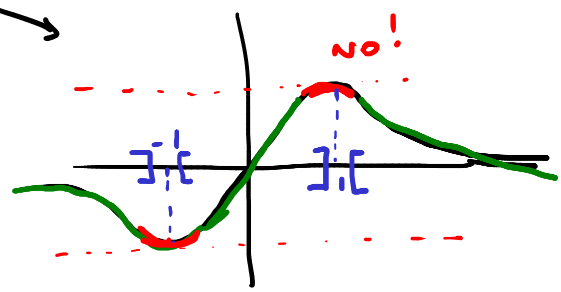
assol: $\mathbb{R} \setminus \{-1, 1\}$ $\rightarrow$



unif.

$]-\infty, a] \cup [b, +\infty[$

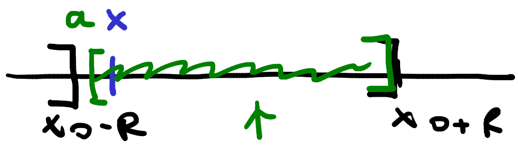
$\forall a < 1 < b$



totale:

$]-\infty, -b] \cup [-a, a] \cup [b, +\infty[, \quad \forall a < 1 < b.$ □

Verifico che f (somma di una serie di potenze) è continua nell' int. di conv. puntuale.



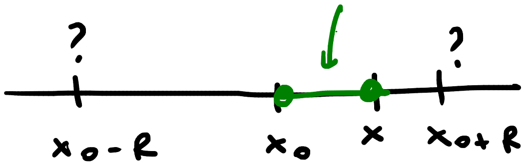
qui la serie conv. unif

$\Rightarrow$ in $[a, x_0+R]$ la somma (funzione f) è continua

$\Rightarrow$ f è cont. in x

chiuso e limitato in cui la serie conv. unif.

$$f(x) = \sum_{n=0}^{\infty} \underbrace{c_n (x-x_0)^n}_{f_n(x)}$$



$$\Rightarrow \int_{x_0}^{x} f(t)\,dt \underset{\text{int. t. a t}}{=} \sum_{n=0}^{+\infty} \int_{x_0}^{x} f_n(t)\,dt$$

funzione integrale di punto iniziale x_0 di f

$$= \sum_{n=0}^{+\infty} \int_{x_0}^{x} c_n (t-x_0)^n\,dt$$

$$= \sum_{n=0}^{+\infty} \left[c_n \frac{(t-x_0)^{n+1}}{n+1} \right]_{x_0}^{x}$$

$$= \sum_{n=0}^{+\infty} c_n \frac{(x-x_0)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{+\infty} \frac{c_n}{n+1} (x-x_0)^{n+1}$$

$$= \sum_{n=1}^{+\infty} \frac{c_{n-1}}{n} (x-x_0)^n$$

$$f(x) = \sum_{n=0}^{+\infty} c_n \underbrace{(x-x_0)^n}_{f_n(x)}$$

$$\sum_{n=0}^{+\infty} f_n'(x) = \sum_{n=1}^{+\infty} c_n\, n \underbrace{(x-x_0)^{n-1}} = \underbrace{\sum_{n=0}^{+\infty} c_{n+1}(n+1)(x-x_0)^n}_{\downarrow}$$

$$\sum_{n=0}^{+\infty} c_n (x-x_0)^n \qquad \sum_{n=1}^{+\infty} \frac{c_{n-1}}{n} (x-x_0)^n$$

$$\sqrt[n]{\left|\frac{c_{n-1}}{n}\right|} = \frac{\sqrt[n]{|c_{n-1}|}}{\sqrt[n]{n}} = \underbrace{\left(\frac{1}{\sqrt[n]{n}}\right)}_{\downarrow\ 1} \cdot \sqrt[n]{|c_{n-1}|}$$

$$\Rightarrow \lim_{n\to+\infty}'' \sqrt[n]{\left|\frac{c_{n-1}}{n}\right|} = \lim_{n\to+\infty}'' \sqrt[n]{|c_{n-1}|}$$

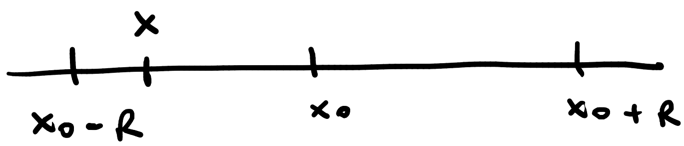
$$\lim_{n\to+\infty}'' \sqrt[n]{|c_n|}$$

$$|c_{n-1}|^{\frac{1}{n}\cdot\frac{n-1}{n-1}}$$

$$= \left(\underbrace{|c_{n-1}|^{\frac{1}{n-1}}}\right) \cdot \frac{n-1}{n} \to 1$$

...

$$f(x) = \sum_{n=0}^{+\infty} c_n (x-x_0)^n \qquad \sum_{n=1}^{+\infty} \underbrace{n c_n (x-x_0)^{n-1}}_{f_n'(x)}$$



$$f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \qquad x \in]x_0 - R_a, x_0 + R_a[$$

$$g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n \qquad x \in]x_0 - R_b, x_0 + R_b[$$

Suppongo che esista $\delta \in \mathbb{R}_+^*$ t.c.

$$f(x) = g(x) \qquad \forall x \in]x_0 - \delta, x_0 + \delta[$$

$$\Rightarrow \quad f(x_0) = g(x_0)$$

$$f'(x_0) = g'(x_0)$$

$$\vdots$$

$$f^{(k)}(x_0) = g^{(k)}(x_0) \qquad \forall k$$

Per $\circledast$:

$$\forall k: \qquad a_k = \frac{f^{(k)}(x_0)}{k!} = \frac{g^{(k)}(x_0)}{k!} \overset{\circledast}{=} b_k \qquad \square$$