17 ottobre 2023

Richiamo sulla dim. del criterio della radice.

$\{x_n\}$ $\qquad \lim_{n\to+\infty} \sqrt[n]{|x_n|} =: L$

$L<1 \qquad L+\varepsilon<1 \qquad \text{dfnt}: \sqrt[n]{|x_n|} < L+\varepsilon$

$\qquad\qquad\qquad\qquad\qquad\qquad " \qquad |x_n| < (L+\varepsilon)^n$

$L>1 \qquad \ldots \qquad \text{freq.} \quad |x_n| > 1$

$\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow |x_n| \not\to 0 \quad \ldots$

Provo a vedere se "funziona" anche per il rapporto.

Sia $\{x_n\}$ con $x_n \neq 0$ (definitivamente)

Pongo $\lim_{n\to+\infty} \frac{|x_{n+1}|}{|x_n|} =: L$

Suppongo $L<1$.

Fisso $\varepsilon \in \mathbb{R}_+^*$ t.c. $L+\varepsilon<1$:

$\text{dfnt}: \frac{|x_{n+1}|}{|x_n|} < L+\varepsilon$

$\Rightarrow \exists \nu \in \mathbb{N}$ t.c. $\forall n \geq \nu$: $\frac{|x_{n+1}|}{|x_n|} < L+\varepsilon$

$|x_{n+1}| < (L+\varepsilon)|x_n|$

$\Rightarrow \quad |x_{\nu+1}| < (L+\varepsilon)|x_\nu|$

$|x_{\nu+2}| < (L+\varepsilon)|x_{\nu+1}| < (L+\varepsilon)^2|x_\nu|$

$\vdots$

$\forall k \geq 1: \quad |x_{\nu+k}| < (L+\varepsilon)^k |x_\nu| \qquad n = \nu+k$

$\Rightarrow \forall n \geq \nu: \quad |x_n| < (L+\varepsilon)^{n-\nu} |x_\nu|$

$$\forall n \geq \nu: \quad |x_n| < \underbrace{(L+\varepsilon)^n}_{\substack{\text{termine di} \\ \text{serie geom. conv.}}} \frac{|x_\nu|}{(L+\varepsilon)^\nu}$$

non dip. da n

$$\Rightarrow \sum_n |x_n| \text{ conv.} \iff \sum_n x_n \text{ conv. assol.}$$

Se suppongo $L > 1$, ragionando come nella dim. del crit. della radice, posso dire che

$$\text{frequent:} \quad \frac{|x_{n+1}|}{|x_n|} > 1$$

$$\text{"} \quad |x_{n+1}| > |x_n|$$

$$\Rightarrow ???$$

non riesco a concludere!

Suppongo $\displaystyle \lim_{n\to+\infty}{}' \frac{|x_{n+1}|}{|x_n|} > 1$

Posso dire che definit.: $\displaystyle \frac{|x_{n+1}|}{|x_n|} > 1$

$$\Rightarrow \exists\, k \in \mathbb{N} \text{ t.c. } \{|x_n|\}_{n \geq k} \text{ strett. cresc.}$$

$$\Rightarrow |x_n| \not\to 0 \Rightarrow x_n \not\to 0$$

$$\Rightarrow \sum_n x_n \text{ non conv.}$$

Quindi:

se $\displaystyle \lim_{n\to+\infty}{}'' \frac{|x_{n+1}|}{|x_n|} < 1$: la serie conv. assol.

se $\displaystyle \lim_{n\to+\infty}{}' \frac{|x_{n+1}|}{|x_n|} > 1$: la serie non converge

(criterio "asimmetrico")

Se $\liminf_{n\to+\infty} \frac{|x_{n+1}|}{|x_n|} \leq 1 \leq \limsup_{n\to+\infty} \frac{|x_{n+1}|}{|x_n|}$

sul carattere della serie non si può dire nulla.

Es: $x_n = \begin{cases} 2^n & n \text{ pari} \\ 3^n & n \text{ dispari} \end{cases}$

$$\frac{x_{n+1}}{x_n} = \begin{cases} \frac{3^{n+1}}{2^n} & n \text{ pari} \\ \frac{2^{n+1}}{3^n} & n \text{ disp.} \end{cases} = \begin{cases} \underbrace{\left(\frac{3}{2}\right)^n}_{\to +\infty} \cdot 3 & n \text{ pari} \\ \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} \cdot 2 & n \text{ disp.} \end{cases}$$

$\Rightarrow \limsup_{n\to+\infty} \frac{x_{n+1}}{x_n} = +\infty > 1, \quad \liminf_{n\to+\infty} \frac{x_{n+1}}{x_n} = 0 < 1$

Oss: $\left.\begin{array}{l} \forall n: \; x_n \geq 2^n \\ \sum_n 2^n \text{ div.} \end{array}\right\} \Rightarrow \sum_n x_n$ div.

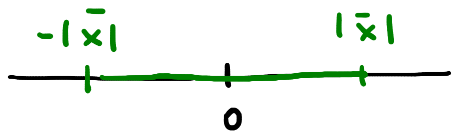
Considero $y_n = \begin{cases} \frac{1}{2^n} & n \text{ pari} \\ \frac{1}{3^n} & n \text{ disp.} \end{cases}$

$$\frac{y_{n+1}}{y_n} = \begin{cases} \frac{1}{3^{n+1}} \cdot 2^n & n \text{ pari} \\ \frac{1}{2^{n+1}} \cdot 3^n & n \text{ disp.} \end{cases} = \begin{cases} \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} \cdot \frac{1}{3} & n \text{ p.} \\ \underbrace{\left(\frac{3}{2}\right)^n}_{\to +\infty} \cdot \frac{1}{2} & n \text{ disp.} \end{cases}$$

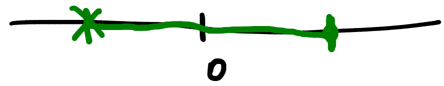
$\Rightarrow \liminf_{n\to+\infty} \frac{y_{n+1}}{y_n} = 0 < 1 < +\infty = \limsup_{n\to+\infty} \frac{y_{n+1}}{y_n}$

Osservo: $\left.\begin{array}{l} 0 < y_n \leq \frac{1}{2^n} \\ \sum_n \frac{1}{2^n} \text{ conv.} \end{array}\right\} \Rightarrow \sum_n y_n$ conv.

Dim. il lemma fondamentale



Suppongo che $\sum_n c_n \bar{x}^n$ converga



$\Rightarrow \{c_n \bar{x}^n\}$ è infinitesima

$\Rightarrow \{c_n \bar{x}^n\}$ è limitata

cioè: $\exists M \in \mathbb{R}_+^*$ tc. $|c_n \bar{x}^n| \leq M \quad \forall n$

Fisso $x \in]-|\bar{x}|, |\bar{x}|[$. Per ogni n:

$$\underline{|c_n x^n|} = \left| c_n \bar{x}^n \cdot \frac{x^n}{\bar{x}^n} \right| = |c_n \bar{x}^n| \left(\frac{|x|}{|\bar{x}|}\right)^n$$

$$\underline{\leq} \; \underline{M \left(\frac{|x|}{|\bar{x}|}\right)^n}$$

$x \in]-|\bar{x}|, |\bar{x}|[\Rightarrow |x| < |\bar{x}| \Rightarrow 0 \leq \frac{|x|}{|\bar{x}|} < 1$

$\Rightarrow \sum_n \left(\frac{|x|}{|\bar{x}|}\right)^n$ conv.

mult. + confr.

$\Rightarrow \sum_n |c_n x^n|$ conv.

cioè: $\sum_n c_n x^n$ conv. assol.

Mi basta provare la conv. totale in compatti del tipo $[-a, a]$, con $0 < a < |\bar{x}|$

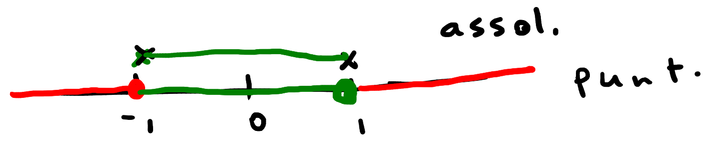
Fissato a come sopra: $\sup_{x \in [-a,a]} |c_n x^n| = |c_n a^n|$

$\sum_n |c_n a^n|$ conv. $(a \in]-|\bar{x}|, |\bar{x}|[)$

$\Rightarrow \sum_n \sup_{x \in [-a,a]} |c_n x^n|$ conv., cioè $\sum_n c_n x^n$ conv.

totalm. in $[-a,a]$ $\square$

$$\sum_n (-1)^n \frac{x^n}{n}$$



r.d.c. $= 1$

Dimostro il teor. sul r.d.c.

$$\text{r.d.c.} \overset{\text{def}}{=} \sup \underbrace{\left\{ x \in \mathbb{R}^n \mid \sum_n c_n x^n \text{ converge} \right\}}_{=:\, A}$$

① $\sup A = 0 \iff$ la serie conv. solo per $x = 0$

$(\Leftarrow)$ la serie conv. solo per $x = 0 \iff A = \{0\}$

$\Rightarrow \sup A = 0$ ✓

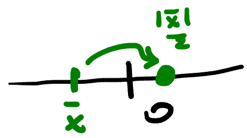
$(\Rightarrow)$ suppongo $\sup A = 0$

$\Rightarrow$ se la serie conv. in x, allora $x \leq 0$

Se la serie convergesse in $\bar{x} < 0$, per il lemma:

la serie convergerebbe assol.

e quindi puntualm. in $]-|\bar{x}|, |\bar{x}|[$,



quindi per esempio anche in $\frac{|\bar{x}|}{2} > 0$, assurdo! ✓

② $\sup A = +\infty \iff \sum_n c_n x^n$ conv. assolut. in $\mathbb{R}$

$(\Leftarrow)$ $\sum_n c_n x^n$ conv. assol. in $\mathbb{R} \Rightarrow$

$\sum_n c_n x^n$ conv. punt. in $\mathbb{R} \Rightarrow A = \mathbb{R}$

$\Rightarrow \sup A = +\infty$ ✓

$(\Rightarrow)$ Suppongo $\sup A = +\infty$

Fisso $x \in \mathbb{R}$; osservo che $|x|$ non è un maggiorante di A, quindi:

$\exists \bar{x} \in A$ t.c. $|x| < \underbrace{\bar{x}}_{>0}$

Cioè: $\exists \bar{x}$ t.c. $\sum_n c_n \bar{x}^n$ converge e $|x| < |\bar{x}|$

$\overset{\text{lemma}}{\Rightarrow}$ la serie conv. assol. in x. $\square$

③ $\sup A = \delta \in]0, +\infty[\iff$ la serie conv. assol. in $]-\delta, \delta[$ e non conv. in $]-\infty, -\delta[\cup]\delta, +\infty[$

($\Leftarrow$)

$\sum_n c_n x^n$ conv. assol. in $]-\delta, \delta[$ $\Rightarrow$

$\sum_n c_n x^n$ conv. punt in $]-\delta, \delta[$ $\Rightarrow$

$]-\delta, \delta[\subset A$ $\Rightarrow$ $\sup A \geq \delta$

Se fosse $\sup A > \delta$, δ non sarebbe maggior. di A

$\Rightarrow$ $\exists \underbrace{\bar{x} \in A}_{\Updownarrow}$ t.c. $\bar{x} > \delta$

la serie conv. in $\bar{x}$

$\Rightarrow$ la serie convergerebbe in $\bar{x} > \delta$, assurdo!

Quindi: $\sup A = \delta$. ✓

($\Rightarrow$) Suppongo $\sup A = \delta$

Fisso $x \in]-\delta, \delta[$, cioè $|x| < \delta = \sup A$

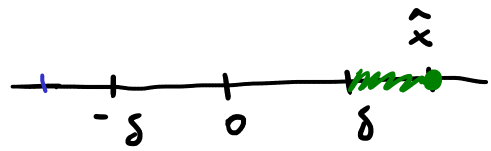
$\Rightarrow$ $\exists \bar{x} \in A$ tc $\bar{x} > |x|$

$\overset{\text{lemma}}{\Rightarrow}$ la serie conv. assolut. in x

Mi resta da provare che la serie $\underline{\text{non}}$ converge se $|x| > \delta$

Per assurdo, suppongo che esista $\hat{x}$ t.c. $|\hat{x}| > \delta$ e la serie conv. in $\hat{x}$

Per il lemma, la serie converge per tutti i punti di $]\delta, |\hat{x}|[$



$\Rightarrow \;]\delta, |\hat{x}|[\, \subset A \;\Rightarrow\; \overbrace{\sup A}^{=\delta} \geq |\hat{x}| > \delta$ assurdo!

$\square$

Es: $n!\, x^n$

$x \neq 0$: $\dfrac{|(n+1)!\, x^{n+1}|}{|n!\, x^n|} = (n+1)\underbrace{|x|}_{>0} \xrightarrow[n \to +\infty]{} +\infty$

$\overset{\text{crit. rapp.}}{\Rightarrow}$ la serie $\underline{\text{non}}$ converge

$\Rightarrow$ r.d.c. $= 0$

Dimostro il crit. di Cauchy-Hadamard.

Oss: $\forall n, \forall x$: $\sqrt[n]{|c_n x^n|} = \sqrt[n]{|c_n||x|^n} = \sqrt[n]{|c_n|}\, |x|$

Pongo $\alpha := \lim_{n \to \infty}{}'' \sqrt[n]{|c_n|}$

Se $\alpha = 0$: $\underline{\forall x \in \mathbb{R}}$:

$\lim_{n \to +\infty}{}'' \sqrt[n]{|c_n x^n|} = \lim_{n \to +\infty}{}'' \left(\sqrt[n]{|c_n|}\, \underline{|x|}\right)$

$$= \left(\lim_{n\to+\infty} \sqrt[n]{|c_n|} \right) |x| = 0$$

crit. radice

$$\Rightarrow \sum_n c_n x^n \quad \text{conv. assol.}$$

Teor. 2

$$\Rightarrow \quad \text{r.d.c.} = +\infty$$

Se $\alpha = +\infty$: $\forall x \neq 0$:

$$\lim_{n\to+\infty} \sqrt[n]{|c_n x^n|} = \dots = \left(\underbrace{\lim_{n\to+\infty} \sqrt[n]{|c_n|}}_{=+\infty} \right) \underbrace{|x|}_{>0} = +\infty$$

crit. radice

$$\Rightarrow \sum_n c_n x^n \quad \text{non conv.}$$

cioè: la serie conv. solo per $x = 0$

Teor. 1

$$\Rightarrow \quad \text{rdc} = 0 .$$

Se $0 < \alpha < +\infty$:

Se $|x| < \frac{1}{\alpha}$:

$|x| < \frac{1}{\alpha}$

$$\lim_{n\to+\infty} \sqrt[n]{|c_n x^n|} = \left(\lim_{n\to+\infty} \sqrt[n]{|c_n|} \right) |x| = \alpha |x| < 1$$

$\Rightarrow$ la serie conv. assol.

Se $|x| > \frac{1}{\alpha}$:

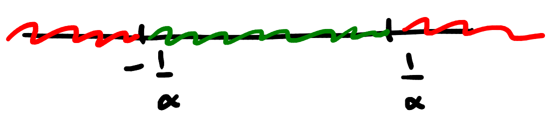
$|x| > \frac{1}{\alpha}$

$$\lim_{n\to+\infty} \sqrt[n]{|c_n x^n|} = \dots = \alpha |x| > 1$$

$\Rightarrow$ la serie non conv.

Teor. 3

$$\Rightarrow \quad \text{r.d.c.} = \frac{1}{\alpha} . \qquad \square$$



Es: $\sum_n (-1)^n \frac{x^n}{n}$

$$c_0 = 0$$

$$c_n = \frac{(-1)^n}{n} \quad \forall n \geq 1$$

$$\sqrt[n]{|c_n|} = \sqrt[n]{\frac{1}{n}} = \frac{1}{\underbrace{\sqrt[n]{n}}_{\to 1}} \longrightarrow 1 =: \alpha$$

$$\Rightarrow \text{rdc} = \frac{1}{1} = 1$$

$\sum_n x^n \qquad c_n = 1 \quad \forall n$

$$\Rightarrow \sqrt[n]{|c_n|} = 1 \quad \forall n \quad \Rightarrow \alpha = 1$$

$$\Rightarrow \text{rdc} = 1$$

$\sum_n \frac{x^n}{n!} \qquad c_n = \frac{1}{n!}$

$$\frac{|c_{n+1}|}{|c_n|} = \frac{1}{(n+1)!} \cdot n! = \frac{1}{n+1} \xrightarrow[n \to +\infty]{} 0 =: \alpha$$

$$\Rightarrow \text{rdc} = +\infty$$

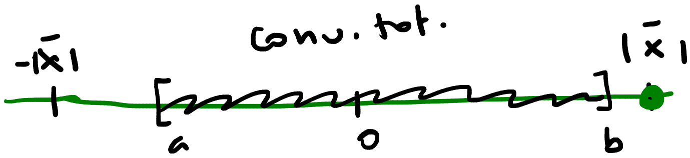
$\sum_n n!\, x^n \qquad c_n = n!$

$$\frac{|c_{n+1}|}{|c_n|} = \frac{(n+1)!}{n!} = n+1 \xrightarrow[n \to +\infty]{} +\infty =: \alpha$$

$$\Rightarrow \text{rdc} = 0$$

Oss. sul caso $R = +\infty$

conv. assol.



$f_n(x) = c_n x^n$, $\quad \forall n \quad \sup_{x \in \mathbb{R}} |f_n(x)| = +\infty$

$\Rightarrow \sup_{\mathbb{R}} |f_n| \not\to 0$

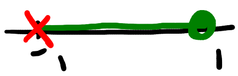
$\Rightarrow \sum_n f_n$ non conv. unif in $\mathbb{R}$

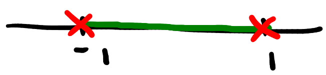
($\Rightarrow$ non conv. tot. in $\mathbb{R}$)

Sulla convergenza negli estremi non si possono fare previsioni:

$\sum_n \frac{(-1)^n}{n} x^n \qquad \qquad \sum_n x^n$

$\swarrow \; R = 1 \; \nearrow$





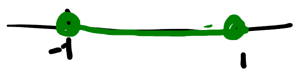
conv. punt.

$\sum_n \frac{x^n}{n^2} \qquad \sqrt[n]{|c_n|} = \sqrt[n]{\frac{1}{n^2}} = \frac{1}{(\sqrt[n]{n})^2} \to 1$

$\Rightarrow R = 1$

$x = 1$, $x = -1$: $\left| \frac{(\pm 1)^n}{n^2} \right| = \frac{1}{n^2}$

$\sum_n \frac{1}{n^2}$ conv. $\Rightarrow$



Oss: $|c_n R^n| = |c_n (-R)^n|$

Supp. $\sum |c_n R^n|$ conv.

$\Rightarrow \sum \sup_{x \in [-R, R]} |c_n x^n|$ conv. $\Rightarrow \sum c_n x^n$ conv. tot. in $[-R, R]$

$$\sum_n (2^n + 3^n)\, x^n \qquad c_n = 2^n + 3^n \quad \forall n$$

$$\sqrt[n]{|c_n|} = \sqrt[n]{2^n + 3^n} = \sqrt[n]{3^n\left(\frac{2^n}{3^n} + 1\right)}$$

$$= 3 \cdot \underbrace{\sqrt[n]{1 + \left(\tfrac{2}{3}\right)^n}}_{\to 1} \longrightarrow 3 \implies R = \frac{1}{3}$$

$$\sum_n (2^n + 3^n)\left(\frac{1}{3}\right)^n = \sum_n \frac{2^n + 3^n}{3^n} \qquad \text{non conv.}$$

$$\frac{2^n + 3^n}{3^n} \sim \frac{3^n}{3^n} \longrightarrow 1$$

$$\sum_n (2^n + 3^n)\left(-\frac{1}{3}\right)^n = \sum_n \underbrace{(-1)^n \underbrace{\frac{2^n + 3^n}{3^n}}_{\to 1,\ \not\to 0}}_{\not\to 0} \qquad \text{non conv.}$$

Conclusione:

conv. punt. e assol. in $\left]-\frac{1}{3}, \frac{1}{3}\right[$

conv. unif. e tot. in $[-a, a]$ $\quad \forall a \in \left]0, \frac{1}{3}\right[$

(in $[a, b]$ $\quad -\frac{1}{3} < a < b < \frac{1}{3}$)

$$\sum_n \frac{(x+1)^n}{(n+1)\,2^n} = \underbrace{\sum_n \frac{1}{(n+1)\,2^n}\,(x+1)^n}_{①} \qquad c_n = \frac{1}{(n+1)\,2^n}$$

$$x_0 = -1$$

Pongo $t = x + 1$ e studio

② $$\sum_n \frac{1}{(n+1)\,2^n}\, t^n \qquad t_0 = 0$$

$$\lim_{n\to+\infty} \sqrt[n]{|c_n|} = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{(n+1)\,2^n}}$$

$$= \lim_{n\to+\infty} \underbrace{\frac{1}{\sqrt[n]{n+1}}}_{\to 1} \cdot \frac{1}{2} = \frac{1}{2} \quad \Rightarrow \quad R = 2$$

$t = 2$: $\quad \sum_n \frac{1}{(n+1)2^n} \cdot 2^n = \sum_n \frac{1}{n+1}$ non conv. (armonica)

$t = -2$: $\quad \sum_n \frac{1}{(n+1)2^n} (-2)^n = \sum_n \frac{(-1)^n}{n+1}$ conv. (Leibniz)

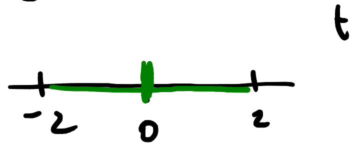
Per ② :

conv. punt. $[-2, 2[$

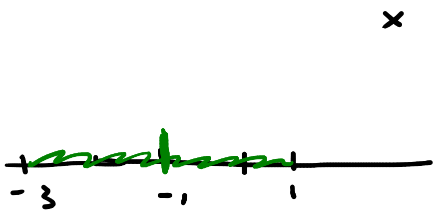
assol. $]-2, 2[$

unif. $[-2, b]$ $\quad -2 < b < 2$

tot. $[a, b]$ $\quad -2 < a < b < 2$



Per ① : $\quad t = x + 1 \qquad x = t - 1$



conv. punt. $[-3, 1[$

assol. $]-3, 1[$

unif. $[-3, \beta]$ $\quad -3 < \beta < 1$

tot. $[\alpha, \beta]$ $\quad -3 < \alpha < \beta < 1$