5 ottobre 2023

- $f_n(x) = \dfrac{x}{1+nx^2}$ $\quad x \in \mathbb{R}, \quad n \in \mathbb{N}$

$\forall n$: $f_n(0) = 0 \implies \exists \lim_{n \to +\infty} f_n(0) = 0 =: f(0)$

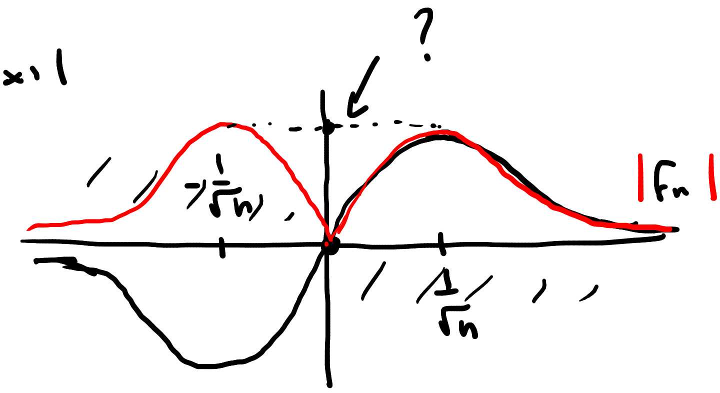
$x \neq 0$: $\lim_n f_n(x) = \lim_n \dfrac{x}{\underbrace{1+nx^2}_{\to +\infty}} = 0 =: f(x)$

$\{f_n\}$ conv. punt. in $\mathbb{R}$ a $f(x) \equiv 0$ $\quad (f'(x) \equiv 0)$

Studio la conv. unif.

Fisso n; $\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = ?$

$= \sup_{x \in \mathbb{R}} |f_n(x)|$



$f_n'(x) = \dfrac{1+nx^2 - x \cdot 2nx}{(1+nx^2)^2}$

$= \dfrac{1-nx^2}{(1+nx^2)^2} \geq 0 \iff |x| \leq \dfrac{1}{\sqrt{n}}$

$\implies \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = f_n\left(\frac{1}{\sqrt{n}}\right) = \dfrac{\frac{1}{\sqrt{n}}}{1 + n \cdot \frac{1}{n}} = \dfrac{1}{2\sqrt{n}}$

$\implies \lim_{n \to +\infty} \sup_{\mathbb{R}} |f_n - f| = \lim_{n \to +\infty} \dfrac{1}{2\sqrt{n}} = 0$

$\implies \{f_n\}$ conv. unif. a $f \equiv 0$

$f_n'(x) = \dfrac{1-nx^2}{(1+nx^2)^2} \xrightarrow[n \to +\infty]{} \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

Quindi:

$$\underbrace{\lim_{n\to+\infty} f_n'(x)}_{\text{non uniforme}} = \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases} \neq f'(x)$$

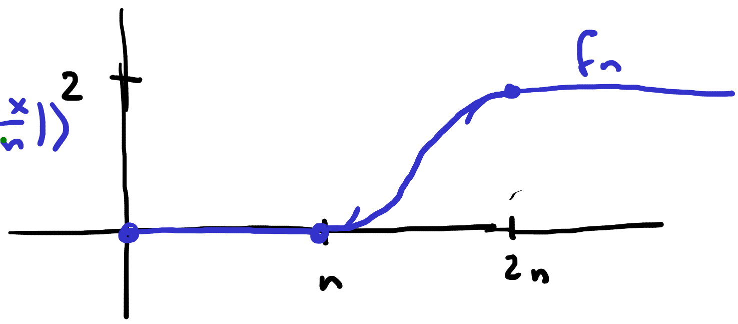
- $f_n(x) = \begin{cases} 0 & x\in[0,n] \\ 1+\cos\left(\frac{\pi x}{n}\right) & x\in]n,2n[\\ 2 & x\in[2n,+\infty[\end{cases}$

$$f_n'(x) = \begin{cases} 0 & x\in[0,n] \\ -\sin\left(\frac{\pi x}{n}\right)\cdot\frac{\pi}{n} & x\in]n,2n[\\ 0 & x\in[2n,+\infty[\end{cases}$$

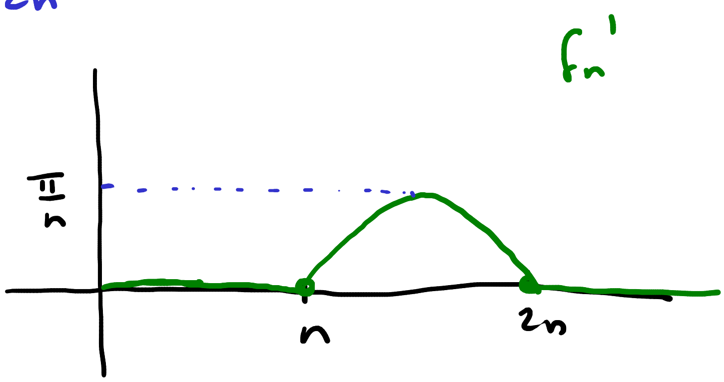
$\lim_{x\to n^-} f_n'(x) = 0$

$\lim_{x\to n^+} f_n'(x) = \lim_{x\to n^+} \left(-\frac{\pi}{n}\sin\left(\frac{\pi x}{n}\right)\right) = 0$

cons. Teor. Lagrange $\Rightarrow \exists f_n'(n) = 0$



Analogamente per $x=2n$



$\sup_{x\in[0,+\infty[} |f_n'(x)| = \frac{\pi}{n} \to 0$

$\Leftrightarrow \{f_n'\}$ conv. unif. in $[0,+\infty[$ a $g\equiv 0$

Osservo che f_n' è continua $\forall n$, quindi $f_n \in C^1$

Studio la convergenza di $\{f_n\}$:

fissato $x \in [0,+\infty[$: dfnt: $n > x$ $\Rightarrow$

dfnt: $x \in [0,n]$ $\Rightarrow$

dfnt: $f_n(x) = 0$

Quindi: $\{f_n(x)\}$ è definitivamente uguale alla succ. costante di valore 0, pertanto:

$$\lim_{n\to+\infty} f_n(x) = 0 =: f(x)$$

Ossia: $\{f_n\}$ conv. punt. in $[0,+\infty[$ a $f \equiv 0$

(Osservo che $f \in C^1([0,+\infty[)$ e $f' = g \ (= \lim_n f_n')$)

Però:

$$\forall n: \sup_{x\in[0,+\infty[} |\underbrace{f_n(x)}_{\geq 0} - \underbrace{f(x)}_{=0}| = \sup_{x\in[0,+\infty[} f_n(x) = 2$$

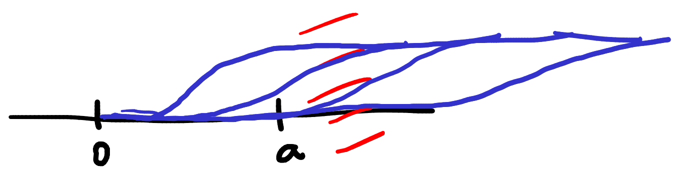
$$\Rightarrow \lim_n \sup_{[0,+\infty[} |f_n - f| = 2 \neq 0$$

Quindi: $\{f_n\}$ <u>non</u> conv. a f unif. in $[0,+\infty[$

Il TPLSSD garantisce che $\{f_n\}$ conv. a f unif. nei compatti di $[0,+\infty[$.

Verifico (anche se è superfluo) che è proprio così:

fisso $a > 0$;



$$\sup_{[0,a]} |f_n - f| \underset{\text{dfnt}}{=} 0 \quad \leftarrow \quad \text{dfnt: } f_n\big|_{[0,a]} \equiv 0$$

$$\Rightarrow \lim_n \sup_{[0,a]} |f_n - f| = 0$$

$\square$

$f \in C^1([a,b],\mathbb{R}) \longmapsto \|f'\|_\infty$ <u>NON è</u> una norma:

$\|f'\|_\infty = 0 \Rightarrow f' \equiv 0 \Rightarrow f \equiv c$

$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$ <u>è</u> una norma:

$$\|f\|_{C^1} = 0 \iff \underbrace{\|f\|_\infty}_{\geq 0} + \underbrace{\|f'\|_\infty}_{\geq 0} = 0$$

$$\Rightarrow \|f\|_\infty = 0 \Rightarrow f \equiv 0 \text{ in } [a,b]$$

$\|\cdot\|_\infty$ è una norma

$$\|\lambda f\|_{C^1} = \|\lambda f\|_\infty + \|\underset{(\lambda f)'}{\lambda f'}\|_\infty$$

$$= |\lambda|\,\|f\|_\infty + |\lambda|\,\|f'\|_\infty = |\lambda|\,\|f\|_{C^1}$$

$$\|f+g\|_{C^1} = \|f+g\|_\infty + \|\underset{(f+g)'}{f'+g'}\|_\infty$$

$$\leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty$$

$$= \|f\|_{C^1} + \|g\|_{C^1}$$

Verifico che $(C^1([a,b],\mathbb{R}), \|\cdot\|_{C^1})$ è sp. di Banach.

$\{f_n\}$ di Cauchy in $(C^1([a,b],\mathbb{R}), \|\cdot\|_{C^1})$

$$\iff \|f_n - f_m\|_{C^1} \xrightarrow[n,m\to+\infty]{} 0$$

$\|h\|_{C^1} = \|h\|_\infty + \|h'\|_\infty$

$$\iff \underbrace{\|f_n - f_m\|_\infty}_{\geq 0} + \underbrace{\|f_n' - f_m'\|_\infty}_{\geq 0} \xrightarrow[n,m\to+\infty]{} 0$$

TCO

$\Rightarrow \quad \|f_n - f_m\|_\infty \xrightarrow[n,m\to+\infty]{} 0 \,, \quad \|f_n' - f_m'\|_\infty \xrightarrow[n,m\to+\infty]{} 0$

$\Leftrightarrow \quad \begin{matrix}\{f_n\} \\ \{f_n'\}\end{matrix}$ di Cauchy in $(\underbrace{C([a,b],\mathbb{R}), \|\cdot\|_\infty}_{\text{completo}})$

$\Rightarrow \quad \{f_n\}, \{f_n'\}$ convergono in $(C([a,b],\mathbb{R}), \|\cdot\|_\infty)$

$\Leftrightarrow \quad \exists \begin{matrix} f \\ g \end{matrix} \in C([a,b],\mathbb{R})$ t.c. $\begin{matrix}\|f_n - f\|_\infty \to 0 \\ \|f_n' - g\|_\infty \to 0\end{matrix} \quad n\to+\infty$ (*)

$\Rightarrow \quad f_n \to f$ unif. in $[a,b]$

$f_n' \to g$ unif. in $[a,b]$

TPLSSD

$\Rightarrow \quad f$ è deriv. e $f' = \underset{\uparrow\ \text{continua}}{g}$

$\Rightarrow \quad f \in C^1([a,b],\mathbb{R})$ e $f' = g$

Riparto da (*) :

$\underbrace{\|f_n - f\|_\infty}_{\to 0} + \underbrace{\|f_n' - g\|_\infty}_{\to 0} \xrightarrow[n\to+\infty]{} 0$

$g = f'$

$\Leftrightarrow \quad \|f_n - f\|_\infty + \|f_n' - f'\|_\infty \xrightarrow[n\to+\infty]{} 0$

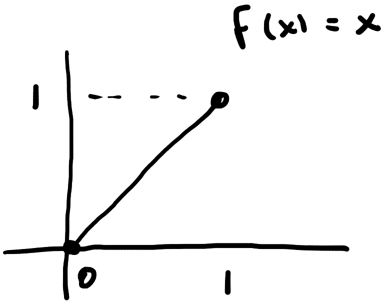
def. di $\|\cdot\|_{C^1}$

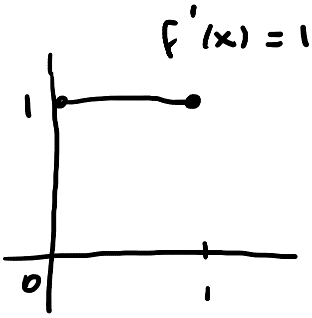
$\Leftrightarrow \quad \|f_n - f\|_{C^1} \xrightarrow[n\to+\infty]{} 0$

Quindi: $\{f_n\}$ conv. a f in $(C^1([a,b],\mathbb{R}), \|\cdot\|_{C^1})$

$\square$

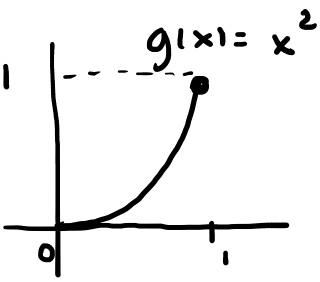
Es:

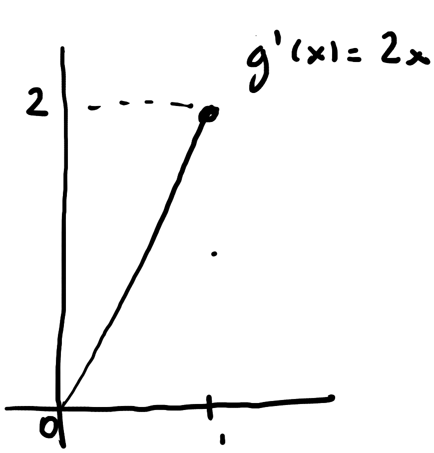




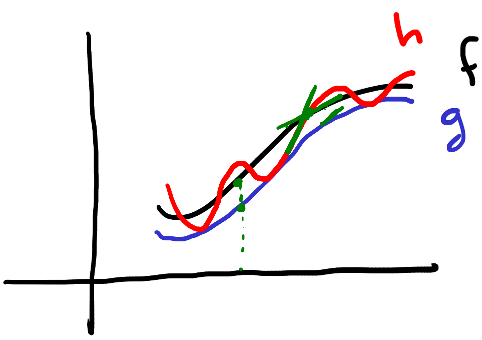
$[a,b] = [0,1]$

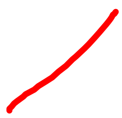
$$\|f\|_{C^1} = \underset{\substack{\| \\ \sup_{[0,1]}|f| = 1}}{\|f\|_\infty} + \underset{\substack{\| \\ 1}}{\|f'\|_\infty} = 2$$





$$\|g\|_{C^1} = \|g\|_\infty + \|g'\|_\infty = 1 + 2 = 3$$





Perché la conv. totale di una serie di funzioni implica la conv. assoluta?

$$\forall \bar{x} \in X : 0 \le \|f_n(\bar{x})\|_Y \le \sup_{x \in X} \|f_n(x)\|_Y$$

Se $\sum_n \sup_{x \in X} \|f_n(x)\|_Y$ converge, allora per il criterio di confronto per serie numeriche a termini non negativi:

$$\sum_n \|f_n(\bar{x})\|_Y \text{ converge.}$$

$\bar{x}$ arbitrario $\Rightarrow$ $\sum_n f_n$ conv. assol. in X.

Es. (conv. assoluta $\not\Rightarrow$ conv. totale)

$$f_n(x) = \underbrace{nxe^{-nx}}_{\geq 0} \qquad x \in [0,+\infty[$$

$\Rightarrow$ per questa serie:

conv. assoluta $\Leftrightarrow$ conv. puntuale

Per $x=0$: $f_n(0)=0 \quad \forall n \Rightarrow \sum_n f_n(0)$ converge

Fissato $x \in]0,+\infty[$, utilizzo il criterio del rapporto:

$$\lim_{n\to+\infty} \frac{|f_{n+1}(x)|}{|f_n(x)|} = \lim_{n\to+\infty} \frac{(n+1)x e^{-(n+1)x}}{nxe^{-nx}}$$

$$= \lim_{n\to+\infty} \frac{n+1}{n} e^{-x} = e^{-x} < 1$$

$x>0$

$\Rightarrow$ la serie di termine $|f_n(x)|$ ($= f_n(x)$) converge.

Conclusione: $\sum_n f_n$ conv. assolutamente in $[0,+\infty[$

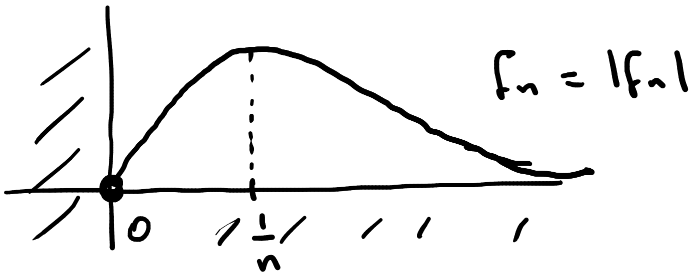
Verifico se converge totalmente:

fisso n e calcolo $\sup_{x\in[0,+\infty[} |f_n(x)|$

nxe^{-nx}

$$f_n'(x) = n\left[e^{-nx} + x\,e^{-nx}(-n)\right]$$

$$= \underbrace{n\,e^{-nx}}_{>0}\;\underbrace{(1-nx)}_{\geq 0 \iff x \leq \frac{1}{n}}$$



$$\forall n:\ \sup_{[0,+\infty[} |f_n| = f_n\left(\tfrac{1}{n}\right) = n\cdot\frac{1}{n}\,e^{-n\cdot\frac{1}{n}} = e^{-1}$$

La serie di termine e^{-1} non converge,

quindi: $\sum_n f_n$ <u>non</u> converge totalmente

in $[0,+\infty[$

Tabella riassuntiva:

$$\begin{array}{ccccc}
 & & & (Y, \|\cdot\|_Y) \text{ Banach} & \\
 & & \text{ASSOLUTA} & \Longrightarrow & \\
\text{TOTALE} & \Longrightarrow & \cancel{\Downarrow}\quad\cancel{\Uparrow} & & \text{PUNTUALE} \\
 & \Longrightarrow & \text{UNIFORME} & \Longrightarrow & \\
 & (Y, \|\cdot\|_Y) \text{ Banach} & & &
\end{array}$$

Riprendo $f_n(x) = n\,x\,e^{-nx}$ $\quad x \in \underline{\underline{\mathbb{R}}}$

Osservo che per $x<0$:

$$\lim_{n\to+\infty} \frac{|f_{n+1}(x)|}{|f_n(x)|} = \dots = e^{-x} > 1$$

$$\Rightarrow \sum_n |f_n(x)| \text{ non converge}$$

$$|nxe^{-nx}| = n\,e^{-nx}|x| = -\,nxe^{-nx}$$

↑ no totale / no unif.

$$\sup_{[0,+\infty[} |f_n| = e^{-1} \not\to 0$$

$$\Rightarrow \sum_n f_n \text{ non conv. unif. in } [0,+\infty[$$