28 settembre 2023

$\langle \cdot , \cdot \rangle : X \to \mathbb{R}$ (sp. vettoriale) prod. scalare

norma associata $\|x\| := \sqrt{\langle x, x\rangle}$

$(X, \|\cdot\|)$ Banach $\overset{def}{\iff}$ $(X, \langle \cdot , \cdot \rangle)$ Hilbert

Es. di metrica non completa in $C([a,b], \mathbb{R})$

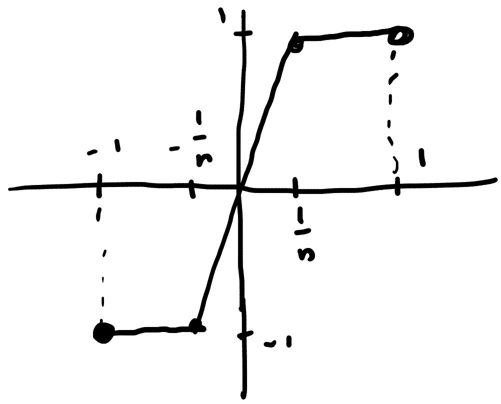
SOSTITUISCE L'ESEMPIO PRECEDENTE

$d_1 : C([-1,1], \mathbb{R}) \times C([-1,1], \mathbb{R}) \to \mathbb{R}$

$$d_1(f,g) := \int_{-1}^{1} |f(x) - g(x)|\, dx$$

d_1 è una metrica (già noto)

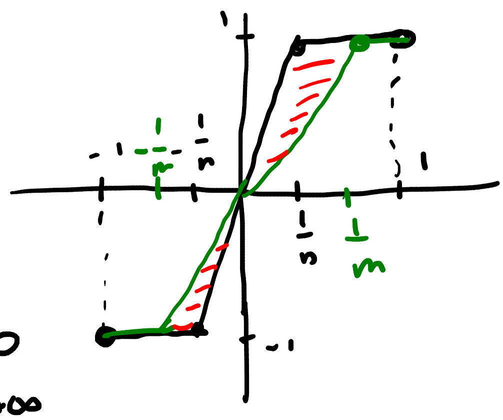
$$f_n(x) = \begin{cases} -1 & x \in [-1, -\frac{1}{n}] \\ nx & x \in]-\frac{1}{n}, \frac{1}{n}[\\ 1 & x \in [\frac{1}{n}, 1] \end{cases}$$



$\forall n: \quad f_n \in C([-1,1], \mathbb{R})$

$n, m, \quad n > m:$

$$d_1(f_n, f_m) \overset{def}{=} \int_{-1}^{1} |f_n(x) - f_m(x)|\, dx$$



$$= 2 \cdot \left| \frac{1}{m} - \frac{1}{n} \right| \cdot 1 \cdot \frac{1}{2} \xrightarrow[n,m \to +\infty]{} 0$$

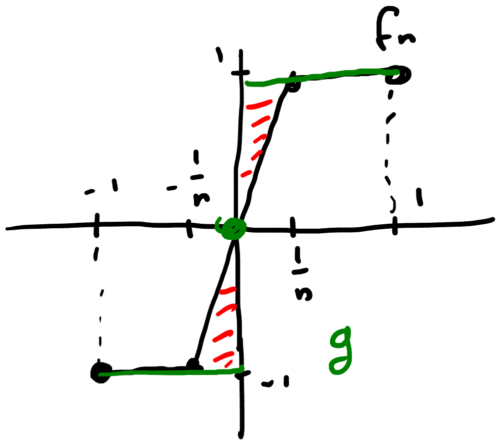
$\Rightarrow \{f_n\}$ è di Cauchy in $(C([-1,1], \mathbb{R}), d_1)$

Oss. $\quad g(x) = \begin{cases} -1 & x \in [-1,0[\\ 0 & x = 0 \\ 1 & x \in]0,1] \end{cases}$

Calcolo

$$\odot \quad \int_{-1}^{1} |f_n(x) - g(x)|\, dx$$

$$= 2 \cdot \frac{1}{n} \cdot 1 \cdot \frac{1}{2} = \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$$



Suppongo che esista $f \in C([-1,1], \mathbb{R})$ t.c.

$$d_1(f_n, f) \xrightarrow[n\to+\infty]{} 0$$

Osservo che:

$$0 \le \int_{-1}^{1} |f(x) - g(x)|\, dx \overset{\forall n}{\le} \underbrace{\int_{-1}^{1} |f(x) - f_n(x)|\, dx}_{= d_1(f_n, f) \ \xrightarrow[n\to+\infty]{} 0 \text{ per ipotesi}} + \underbrace{\int_{-1}^{1} |f_n(x) - g(x)|\, dx}_{\odot \ \xrightarrow[n\to+\infty]{} 0}$$

Quindi:

$$\int_{-1}^{1} |f(x) - g(x)|\, dx = 0$$

cioè:

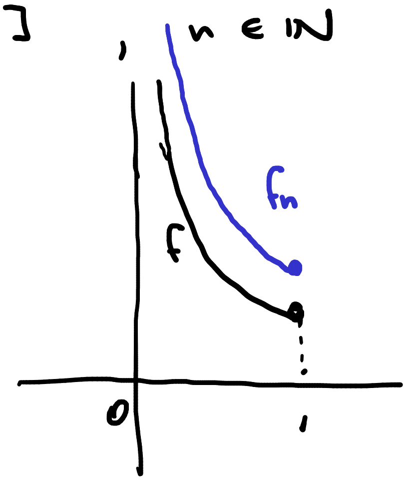
$$\int_{-1}^{0} \underbrace{|f(x) - (-1)|}_{\text{cont. } \ge 0}\, dx = 0 \quad \text{e} \quad \int_{0}^{1} \underbrace{|f(x) - 1|}_{\text{cont. } \ge 0}\, dx = 0$$

$$\Rightarrow \begin{cases} f(x) = -1 & \forall x \in]-1,0[\\ f(x) = 1 & \forall x \in]0,1[\end{cases}$$

incompatibile con la continuità di f! $\square$

Es. $f_n(x) = \frac{1}{x} + \frac{1}{n+1} \qquad x \in \,]0,1]\,, \quad n \in \mathbb{N}$

$f(x) = \frac{1}{x} \qquad x \in \,]0,1]$



f_n, f **non** sono limitate in $]0,1]$; però:

$$\lim_{n\to+\infty} \sup_{x\in]0,1]} |f_n(x) - f(x)| =$$

$$\lim_{n\to+\infty} \sup_{x\in]0,1]} \left| \cancel{\frac{1}{x}} + \frac{1}{n+1} - \cancel{\frac{1}{x}} \right| = \lim_{n\to+\infty} \frac{1}{n+1} = 0$$

Oss. conv. unif $\Rightarrow$ conv. punt.

$\forall x \in X$:

$$0 \le d_Y(f_n(x), f(x)) \le \underbrace{\sup_{z\in X} d_Y(f_n(z), f(x))}_{\to 0 \ (\text{conv. unif.})}$$

TCO: $d_Y(f_n(x), f(x)) \to 0$

Es. di succ. che converge puntualmente ma non uniformemente

$$\forall n: \quad f_n(x) = \frac{e^{nx}}{e^{nx}+1} \qquad x \in \mathbb{R}$$

Fisso $x \in \mathbb{R}$:

$$\lim_{n\to+\infty} e^{nx} = \begin{cases} 0 & x<0 \\ 1 & x=0 \\ +\infty & x>0 \end{cases}$$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{e^{nx}}{e^{nx}+1} = \begin{cases} 0 & x\in]-\infty,0[\\ \frac{1}{2} & x=0 \\ 1 & x\in]0,+\infty[\end{cases}$$

$\{f_n\}$ converge puntualmente in $\mathbb{R}$ a

$$f(x) := \begin{cases} 0 & x\in]-\infty,0[\\ 1/2 & x=0 \\ 1 & x\in]0,+\infty[\end{cases}$$

Fisso n e calcolo

$$\sup_{x\in\mathbb{R}} |f_n(x)-f(x)| =$$

$$\max\Big\{ \sup_{x\in]-\infty,0[} |f_n(x)-0| ,\ \overset{=0}{|f_n(0)-f(0)|} ,\ \sup_{x\in]0,+\infty[} |f_n(x)-1| \Big\}$$

Rappresento graficamente $\{f_n\}$, per $n\geq 1$:

$$\underset{\substack{\uparrow \\ \text{strett.} \\ \text{cresc.}}}{f_n(x)} = \frac{e^{nx}}{1+e^{nx}} = 1 - \frac{1}{1+e^{nx}}$$

str. decr.

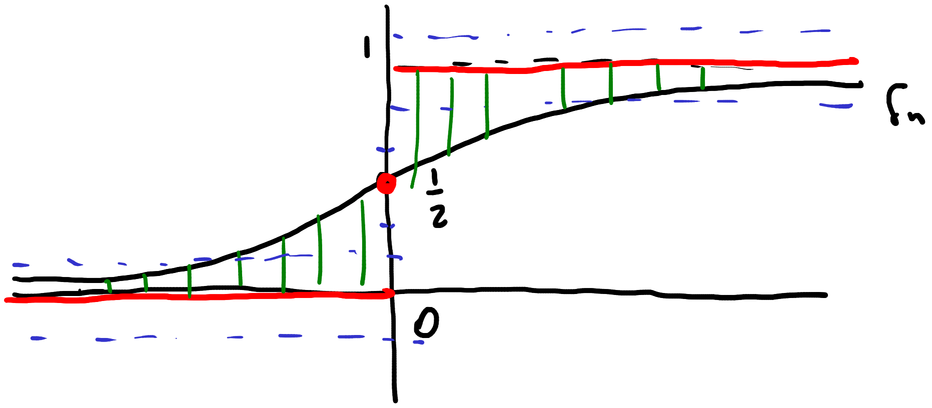
str. cr.

str. cr.

$\forall x:\ f_n(x)>0$

$$\lim_{x\to-\infty} f_n(x) = \lim_{x\to-\infty} \frac{e^{nx}}{1+e^{nx}} = 0$$

$$\lim_{x\to+\infty} f_n(x) = \lim_{x\to+\infty} \frac{\overbrace{e^{nx}}^{\to+\infty}}{\underbrace{1+e^{nx}}_{\to+\infty}} = 1$$



$\forall n \geq 1$:

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to+\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \frac{1}{2} \neq 0 \;!$$

richiesta di chiarimento ...

$\{f_n\}$ conv. unif. a f in X $\overset{def}{\Leftrightarrow}$

$$\lim_{n\to+\infty} \sup_{x\in X} d_Y(f_n(x), f(x)) = 0 \Leftrightarrow$$

$$\forall \varepsilon \in \mathbb{R}_+^* \;\exists \nu \in \mathbb{N} \text{ t.c. } \forall n \geq \nu : \underbrace{\sup_{x\in X} d_Y(f_n(x), f(x)) < \varepsilon}$$

$$\Rightarrow \qquad \Rightarrow d_Y(f_n(x), f(x)) < \varepsilon \quad \forall x \in X$$

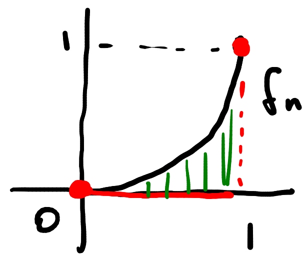
Es: $f_n(x) = x^n \quad \forall x \in [0,1], \quad n \geq 1$

$$f_n(x) \longrightarrow \left.\begin{cases} 0 & x \in [0,1[\\ 1 & x = 1 \end{cases}\right\} =: f(x)$$

Fisso n; $\quad \sup_{x\in[0,1]} |f_n(x)-f(x)| = 1$

$\Rightarrow \lim_n \sup |f_n - f| = 1 \neq 0$

no conv. unif.



Oss:

$$|f_n(x)-f(x)| = \begin{cases} 0 & x\in\{0,1\} \\ x^n & x \in]0,1[\end{cases}$$

Fisso $\varepsilon \in \mathbb{R}_+^*$; mi chiedo quando è soddisfatta la disuguaglianza

$$(*) \qquad |f_n(x)-f(x)| < \varepsilon$$

Se $x\in\{0,1\}$: $(*)$ vale $\forall n$

Se $x \in]0,1[$: $(*) \iff x^n < \varepsilon$

Se $\varepsilon \geq 1$: $(*)$ vera $\forall n$

Se $\varepsilon \in]0,1[$: $(*) \iff \ln(x^n) < \ln(\varepsilon)$

$$\iff n \underbrace{\ln x}_{<0} < \underbrace{\ln \varepsilon}_{<0} \qquad x\in]0,1[$$

$$\iff n > \frac{\ln \varepsilon}{\ln x}$$

$$\iff n \geq \left\lfloor \frac{\ln \varepsilon}{\ln x} \right\rfloor + 1 =: \nu_{\varepsilon,x}$$

$$\lim_{x\to 1^-} \nu_{\varepsilon,x} = +\infty$$

Fisso $a \in]0,1[$ e considero $[0,a]$

Definisco $\nu_\varepsilon := \left\lfloor \frac{\ln \varepsilon}{\ln a} \right\rfloor + 1$

$\forall x \in]0,a]$: $\quad \ln x \leq \ln a \Rightarrow$

$$\frac{1}{\ln x} \geq \frac{1}{\ln a} \Rightarrow$$

$$\frac{\ln \varepsilon}{\ln x} \leq \frac{\ln \varepsilon}{\ln a} \Rightarrow$$

$$\left\lfloor \frac{\ln \varepsilon}{\ln x} \right\rfloor + 1 \leq \left\lfloor \frac{\ln \varepsilon}{\ln a} \right\rfloor + 1$$

$$\nu_{\varepsilon, x} \leq \nu_\varepsilon$$

Pertanto:

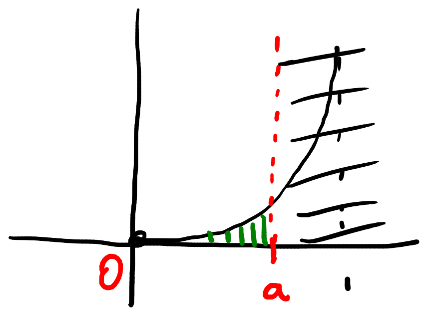
$n \geq \nu_\varepsilon \Rightarrow n \geq \nu_{\varepsilon,x} \quad \forall x \in]0,a]$

$\Rightarrow \circledast$ vale $\forall x \in]0,a]$

$\circledast$ vale anche per $x=0$

$\Rightarrow \circledast$ vale $\forall x \in [0,a]$ $\square$

Con i grafici:



$$\sup_{x \in [0,a]} |f_n(x) - f(x)| = a^n \xrightarrow[n \to \infty]{} 0$$

$0 < a < 1$

Dimostro la prop. "conv. unif. e limitatezza".

Ip: $\{f_n\}$ conv. unif. a f, $\quad f_n$ limitata $\forall n$.

Tesi: f è limitata

Fisso $\varepsilon = 1$.

Dato che $\{f_n\}$ conv. unif. a f, esiste $\nu \in \mathbb{N}$ t.c.

$\forall n \geq \nu$: $\quad d_Y(f_n(x), f(x)) < 1 \quad \forall x \in X$

In particolare: $\quad d_Y(f_\nu(x), f(x)) < 1 \quad \forall x \in X \quad \odot$

Dato che f_ν è limitata, esistono $\bar{y} \in Y$ e $r \in \mathbb{R}_+^*$ t.c.

$$f_\nu(X) \subseteq \bar{B}_r(\bar{y}),$$

cioè:

$$d_Y(f_\nu(x), \bar{y}) \leq r \qquad \forall x \in X \qquad (\bullet\bullet)$$

Per ogni $x \in X$:

$$d_Y(f(x), \bar{y}) \underset{\substack{\text{dis.}\\ \text{tr.}}}{\leq} \underbrace{d_Y(f(x), f_\nu(x))}_{<1 \ \odot} + \underbrace{d_Y(f_\nu(x), \bar{y})}_{\leq r \ (\bullet\bullet)}$$

$$< 1 + r \leq 1 + r =: \tilde{r}$$

Quindi: $\qquad f(X) \subseteq \bar{B}_{\tilde{r}}(\bar{y})$

cioè f è limitata. $\square$

Es. di succ. di funzioni limitate che convergono <u>non</u> uniformemente a una funz. limitata.

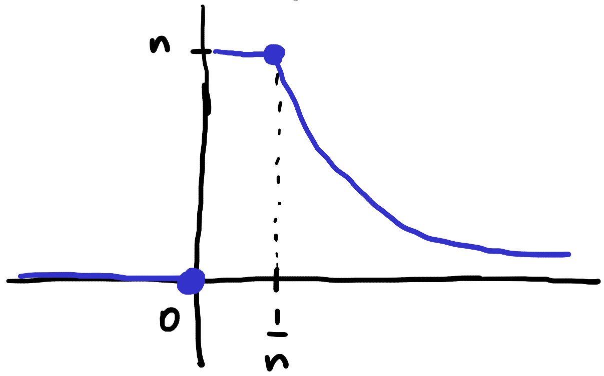
$f_n(x) = x^n \qquad x \in [0,1]$

$$f(x) = \begin{cases} 0 & x \in [0,1[\\ 1 & x = 1 \end{cases}$$

$f_n(x) = \dfrac{e^{nx}}{1 + e^{nx}} \qquad x \in \mathbb{R}$

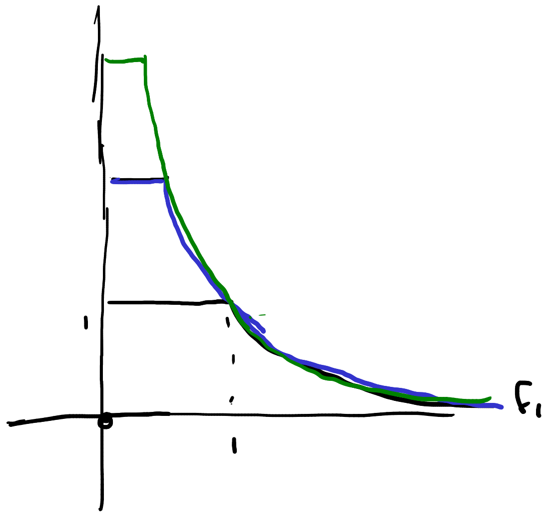
$$f(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$$

Es. di succ. di funzioni limitate che conv. punt. (e non unif.) a una funz. <u>non</u> limitata.



$$f_n(x) = \begin{cases} 0 & x \in]-\infty, 0] \\ n & x \in]0, \frac{1}{n}] \\ \frac{1}{x} & x \in]\frac{1}{n}, +\infty[\end{cases}$$

$\forall n$: f_n è limitata $\quad (f_n(\mathbb{R}) \subset [0, n])$



Fisso $x \in \mathbb{R}$.

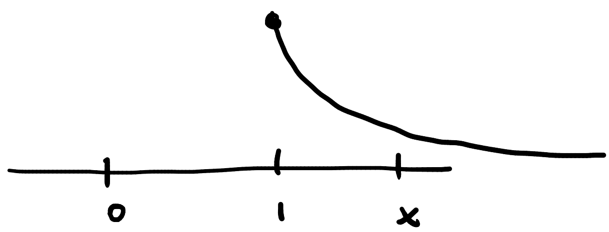
Se $x \in]-\infty, 0]$: $\quad f_n(x) = 0 \quad \forall n$

$\Rightarrow \{f_n(x)\}$ è la succ. costante di valore 0

$\Rightarrow \lim_n f_n(x) = 0 =: f(x)$

Se $x \in]1, +\infty[$:

$f_n(x) = \frac{1}{x} \quad \forall n$



$\Rightarrow \{f_n(x)\}$ è la succ. cost. di valore $\frac{1}{x}$

$\Rightarrow \lim_n f_n(x) = \frac{1}{x} =: f(x)$

Se <u>fisso</u> $x \in]0,1[$:

$f_1(x) = 1$

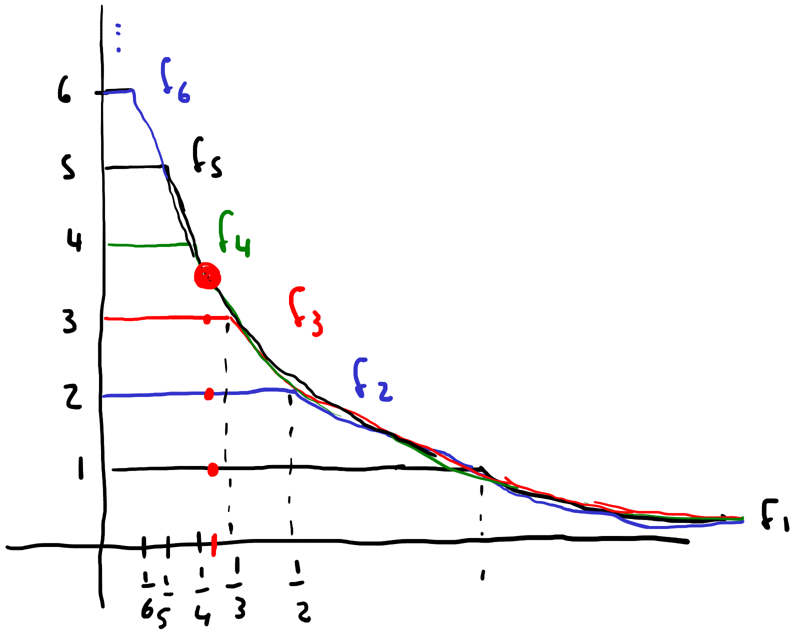
$f_2(x) = 2$

$f_3(x) = 3$

$f_4(x) = \frac{1}{x}$

$f_5(x) = \frac{1}{x}$

$\vdots$

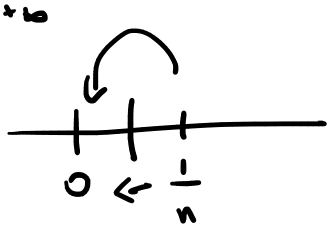


Formalizzo:

fissato $x \in]0,1[$, dato che $\frac{1}{n} \xrightarrow[n\to+\infty]{} 0$,

posso dire che

$$\frac{1}{n} < x \quad \text{definitivamente}$$



$\iff$ dfnt : $x \in]\frac{1}{n}, +\infty[$

$\implies$ dfnt : $f_n(x) = \frac{1}{x}$

Quindi: la succ. $\{f_n(x)\}$ è definitivamente uguale alla succ. costante di valore $\frac{1}{x}$

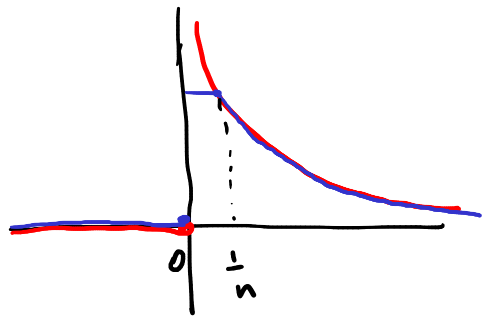
$$\implies \lim_n f_n(x) = \frac{1}{x} =: f(x)$$

Funzione limite puntuale:

$$f(x) = \begin{cases} 0 & x \in]-\infty, 0] \\ \frac{1}{x} & x \in]0, +\infty[\end{cases}$$

NON limitata !

Verifico (non ce n'è bisogno) che $\{f_n\}$ non conv. unif. a f.

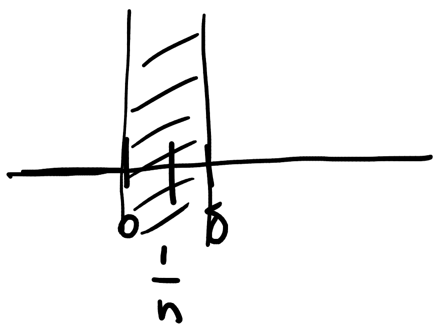


Fisso n:

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$$

$$= \sup_{x \in]0, \frac{1}{n}[} |f_n(x) - f(x)|$$

$$= \sup_{x \in]0, \frac{1}{n}[} \left| n - \frac{1}{x} \right| = \sup_{x \in]0, \frac{1}{n}[} \left(\frac{1}{x} - n \right) = +\infty \; !!$$



dfnt.

$$\sup_{x \in [\delta, +\infty[} |f_n(x) - f(x)| = 0$$

$\uparrow$

$\forall n$ t.c. $\frac{1}{n} < \delta$

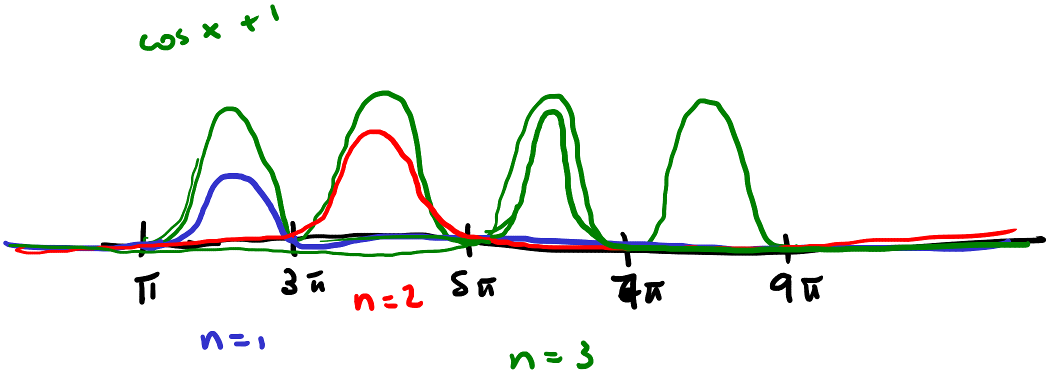
$n > \frac{1}{\delta}$

$\Rightarrow$ conv. unif in $[\delta, +\infty[$

Es. di succ. di funz. continue che conv. punt. e non unif. a una funz. continua.

$$f_n(x) = \begin{cases} \dfrac{n(\cos(x) + 1)}{n+1} & x \in [(2n-1)\pi, (2n+1)\pi] \\ 0 & \text{altrimenti} \end{cases}$$

$n \geq 1$



$f(x) \equiv 0$ continua

$$\sup_{x \in \mathbb{R}} |f_n(x) - \underbrace{f(x)}_{0}| = \frac{n}{n+1} 2 \xrightarrow[n \to +\infty]{} 2 \neq 0$$

□