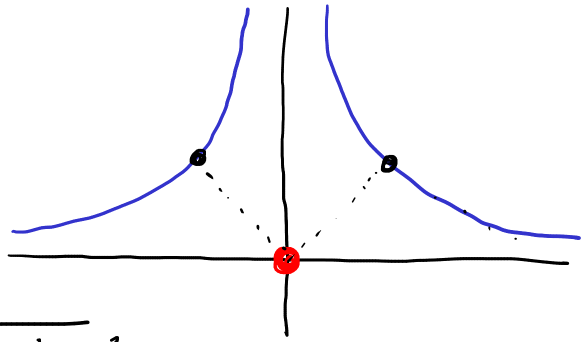
6 giugno 2025

Es. Det. la minima distanza tra l'origine e l'insieme

$$D = \{(x,y) \mid x^2 y = 16\}$$



? $\min_D d$

$d(x,y) := \|(x,y) - (0,0)\| = \sqrt{x^2+y^2}$

Equivale a $\min_D f$, con $f(x,y) = x^2+y^2$

di classe C^1

Oss: $f|_D(x,y) = f\left(x, \frac{16}{x^2}\right) = x^2 + \frac{16^2}{x^4} \xrightarrow[x \to 0]{x \to \pm\infty} +\infty$

$\Rightarrow \nexists \max_D f$

(non mi sorprende, perché D non è compatto, quindi il teor. di Weierstrass non si applica)

Determino i candidati punti di estremo (locale) sul vincolo tramite il TML.

Punti singolari?

$D = \{(x,y) \mid \underbrace{x^2y - 16}_{g(x,y)} = 0$

$$\begin{cases} g_x = 0 \\ g_y = 0 \\ g = 0 \end{cases} \iff \begin{cases} 2xy = 0 \\ x^2 = 0 \\ x^2y - 16 = 0 \end{cases} \qquad \begin{matrix} x = 0 \\ -16 = 0 \end{matrix} \quad !!!$$

Non ci sono punti singolari.

Cerco i punti stazionari di

$$L(x,y,\lambda) := f(x,y) - \lambda g(x,y)$$
$$= x^2+y^2 - \lambda(x^2y-16)$$

$$\begin{cases} L_x = 0 \\ L_y = 0 \\ L_\lambda = 0 \end{cases} \qquad \begin{cases} 2x - \lambda 2xy = 0 \\ 2y - \lambda x^2 = 0 \\ -(x^2y-16) = 0 \end{cases} \qquad \begin{matrix} 2x(1-\lambda y) = 0 \\ \dots \\ \dots \end{matrix}$$

(1) $\begin{cases} x = 0 \\ y = 0 \\ 16 = 0 \quad !! \end{cases}$ $\qquad \begin{cases} 1-\lambda y = 0 \\ \\ \end{cases}$ $\qquad \lambda = \frac{1}{y} \quad (y \neq 0)$

$$\begin{cases} 2y - \frac{1}{y}x^2 = 0 \\ x^2y = 16 \end{cases}$$

$$\begin{cases} 2y^2 = x^2 \\ 2y^3 = 16 \end{cases} \qquad \begin{matrix} x = \pm\sqrt{2}y \\ y = 2 \end{matrix}$$

Candidati punti di estremo:

$$(2\sqrt{2}, 2) \quad e \quad (-2\sqrt{2}, 2)$$

$$f(2\sqrt{2},2) = f(-2\sqrt{2},2) = 8+4 = 12$$

Dalla "geometria" del problema, deduco che

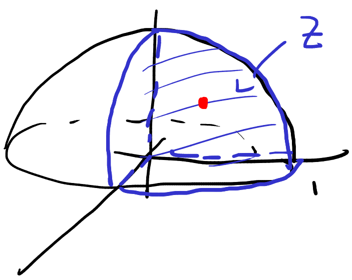
$$12 = \min_D f$$

$$\Rightarrow \quad \min_D d = \sqrt{12}.$$

Esempio (vincolo in $\mathbb{R}^3$ con parametrizzazione)

Determinare gli estremi di: $f(x,y,z) = xyz$

in

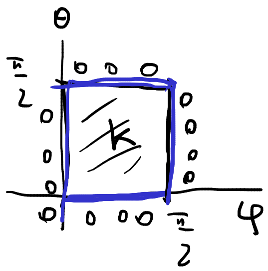


$$\sigma(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$(\varphi,\theta) \in \left[0,\frac{\pi}{2}\right] \times \left[0,\frac{\pi}{2}\right] =: K$$

$$h(\varphi,\theta) := f(\sigma(\varphi,\theta)) = \sin\varphi\cos\theta \cdot \sin\varphi\sin\theta \cdot \cos\varphi$$

$$= \sin^2\varphi\cos\varphi\,\cos\theta\sin\theta \qquad (\geq 0)$$



$$h\big|_{\partial K} \equiv 0 \qquad h\big|_{\mathring{K}} > 0$$

$$\Rightarrow \quad 0 = \min_K h = \min_Z f$$

$\Big($oss: Z è compatto $\Rightarrow$ $\exists \min\limits_Z f$, $\max\limits_Z f\Big)$

Studio h in $\mathring{K}$:

$$\begin{cases} h_\varphi = 0 \\ h_\theta = 0 \end{cases} \qquad \begin{cases} (2\sin\varphi\cos^2\varphi - \sin^3\varphi)\overbrace{\cos\theta\sin\theta}^{\neq 0} = 0 \\ \underbrace{\sin^2\varphi\cos\varphi}_{\neq 0}(-\sin^2\theta + \cos^2\theta) = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \overbrace{\sin\varphi}^{\neq 0}(2\cos^2\varphi - \sin^2\varphi) = 0 \\ -\sin^2\theta + \cos^2\theta = 0 \end{cases}$$

1° ottante

$$2\cos^2\varphi - 1 + \cos^2\varphi = 0 \qquad 3\cos^2\varphi = 1 \qquad \cos\varphi = \frac{1}{\sqrt{3}}$$

Sol. $\varphi = \varphi_0$ con

$\cos\varphi_0 = \frac{1}{\sqrt{3}}$, $\sin\varphi_0 = \sqrt{\frac{2}{3}}$

$$\sin^2\theta = \cos^2\theta \qquad \theta = \frac{\pi}{4}$$

Unico punto staz: $\left(\varphi_0, \frac{\pi}{4}\right)$

$$\Rightarrow \quad \max_K h = \max_{\mathring{K}} h = h\left(\varphi_0, \frac{\pi}{4}\right) = \frac{2}{3}\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{3\sqrt{3}}$$

Conclusione:

$$\max_Z f = \frac{1}{3\sqrt{3}}$$

assunto in $\sigma\left(\varphi_0, \frac{\pi}{4}\right) = \left(\sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2}}, \sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Esempi (estremi vincolati in $\mathbb{R}^3$)

Estremi di $f(x,y,z) = x^2+y^2+z^2$ in

$$Z = \{ (x,y,z) \mid \underbrace{z^2 - xy - 1}_{g(x,y,z)} = 0 \}$$

Già visto: Z non ha punti singolari

Lagrangiana:

$$L(x,y,z,\lambda) = x^2+y^2+z^2 - \lambda(z^2 - xy - 1)$$

Punti staz:

$$\begin{cases} L_x = 0 \\ L_y = 0 \\ L_z = 0 \\ L_\lambda = 0 \end{cases} \qquad \begin{cases} 2x + \lambda y = 0 \\ 2y + \lambda x = 0 \\ 2z - \lambda \cdot 2z = 0 \\ -(z^2 - xy - 1) = 0 \end{cases} \qquad 2z(1-\lambda) = 0$$

1°: $z = 0$

$$\begin{cases} 2x + \lambda y = 0 \\ 2y + \lambda x = 0 \\ xy + 1 = 0 \end{cases} \qquad \begin{cases} 2(x-y) - (x-y)\lambda = 0 \\ 2y + \lambda x = 0 \\ xy + 1 = 0 \end{cases}$$

$$\begin{cases} (x-y)(2-\lambda) = 0 \\ \dots \\ \dots \end{cases}$$

$x = y$, $x^2 + 1 = 0$!!

$$\begin{cases} \lambda = 2 \\ 2y + 2x = 0 \\ -x^2 + 1 = 0 \end{cases} \qquad y = -x$$

Candidati $(1,-1,0)$ $(-1,1,0)$

2°: $\lambda = 1$

$$\begin{cases} 2x + y = 0 \\ 2y + x = 0 \\ z^2 = xy + 1 \end{cases} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x = 0 \\ y = 0 \end{matrix}$$

$z^2 = 1$

Candidati: $(0,0,1)$ $(0,0,-1)$

Valuto f: $f(x,y,z) = x^2+y^2+z^2$

$$f(\pm 1, \mp 1, 0) = 2$$

$$f(0,0,\pm 1) = 1$$

Conclusione:

$$1 = \min_{Z} f$$ Ok!

$$\cancel{2 = \max_{Z} f}$$!!!

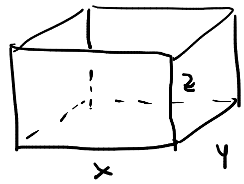
Oss. $f|_Z(x,y,z) = x^2+y^2+\underbrace{xy}_{|xy| \le \frac{x^2+y^2}{2}}+1 =: h(x,y)$

$\|(x,y)\| \to +\infty$: $h(x,y) \to +\infty$

$$\Rightarrow \nexists \max_{Z} f$$

$$f|_Z(\quad) = \underbrace{x^2+y^2+xy}+1 \ge 1$$

- $\max\limits_{Z} V(x,y,z)$



con $V(x,y,z) = xyz$ $(x,y,z>0)$

$$Z := \{ (x,y,z) \mid \underbrace{xy + 2xz + 2yz - 12}_{g(x,y,z)} = 0 \}$$

Da verificare: $\nexists$ punti singolari

Candidati punti di estremo:

$$L(x,y,z,\lambda) = xyz - \lambda(xy + 2xz + 2yz - 12)$$

Punti staz:

$$\begin{cases} yz - \lambda y - 2\lambda z = 0 \\ xz - \lambda x - 2\lambda z = 0 \\ xy - 2\lambda x - 2\lambda y = 0 \\ xy + 2xz + 2yz - 12 = 0 \end{cases}$$

....

$\left(\inf_{z} f = 0\right)$

$\left[(2,2,1) , \quad V_{max} = 4 \right]$

PER I DETTAGLI E PER L'ESEMPIO FINALE CONSULTARE IL TESTO "ANALISI MATEMATICA 2"

DI ENRICO GIUSTI