5 giugno 2025

## Esempio (vincolo parametrizzabile)

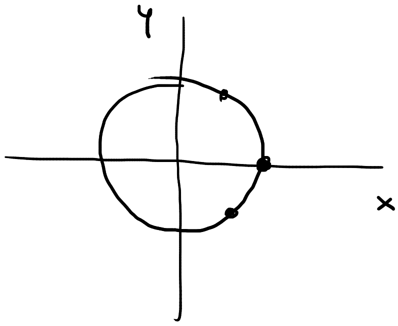
- Determinare gli estremi locali e globali di

$$f(x,y) = 2x^2 + y^2 - x \quad \text{su} \quad S^1 = \{(x,y) \mid x^2+y^2-1=0\}$$

$S^1 : \quad r(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$h : [0,2\pi] \to \mathbb{R}$ t.c. $h(t) = f(r(t)) = 2\cos^2 t + \sin^2 t - \cos t$

$h(0) = 1 = h(2\pi) \; (= f(1,0))$

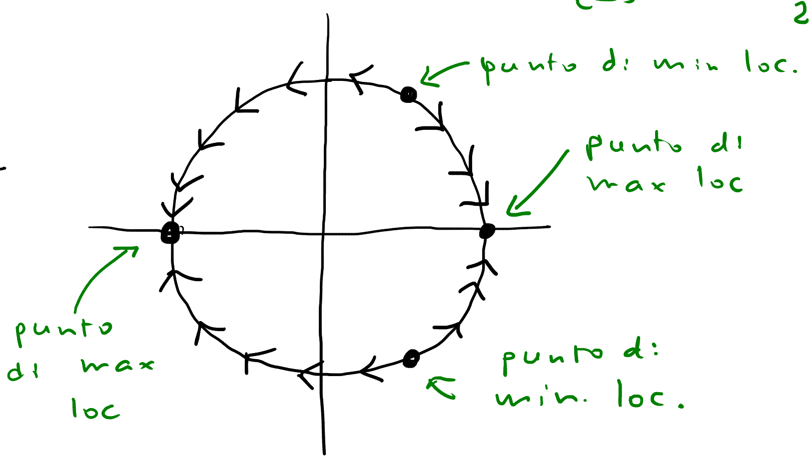



Cerco i punti staz. di $h$!

$h'(t) = -4\cos t \sin t + 2\sin t \cos t + \sin t$

$= -2\cos t \sin t + \sin t = \sin t\,(\underbrace{1 - 2\cos t}_{=0 \iff \cos t = \frac{1}{2}})$

| | $0$ | | $\frac{\pi}{3}$ | | $\pi$ | | $\frac{5}{3}\pi$ | | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\sin t$ | | + | | + | | − | | − | |
| $1-2\cos t$ | | − | | + | | + | | − | |
| $h'(t)$ | | − | | + | | − | | + | |
| $h$ | | ↘ | | ↗ | | ↘ | | ↗ | |




$h\left(\frac{\pi}{3}\right) = f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 2 \cdot \frac{1}{4} + \frac{3}{4} - \frac{1}{2} = \frac{3}{4}$

$h\left(\frac{5}{3}\pi\right) = f\left(+\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \dots = \frac{3}{4}$ $= \min_{S^1} f$

$h(0) = h(2\pi) = f(1,0) = 2 - 1 = 1$ (max loc.)

$h(\pi) = f(-1,0) = 2 + 1 = 3 = \max_{S^1} f$

- Utilizzo il punto precedente per determinare gli estremi globali di $f$ in

$$D = \{(x,y) \mid x^2+y^2 \leq 1\}$$

Oss: $\partial D = S^1$

$$\overset{\text{punto prec.}}{\Rightarrow} \quad \max_{\partial D} f = 3 \quad (= f(-1,0))$$

$$\min_{\partial D} f = \frac{3}{4} \quad \left(= f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)\right)$$

Cerco i candidati punti di estremo in $\mathring{D}$.

Calcolo $f_x(x,y) = 4x-1$, $f_y(x,y) = 2y$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \iff \begin{cases} 4x-1=0 \\ 2y=0 \end{cases} \qquad \left(\frac{1}{4}, 0\right)$$

Superfluo:

$$H_f\left(\frac{1}{4},0\right) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

autoval. $> 0$

$\Rightarrow$ punto di min.loc.

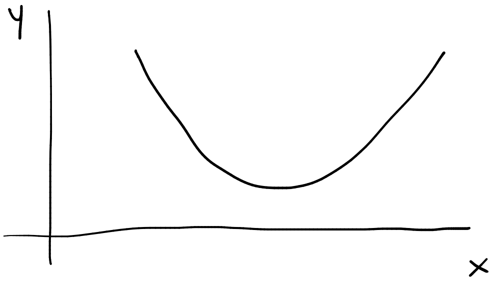
$$f\left(\frac{1}{4},0\right) = 2\cdot\frac{1}{16} + 0 - \frac{1}{4} = -\frac{1}{8} < \frac{3}{4}$$

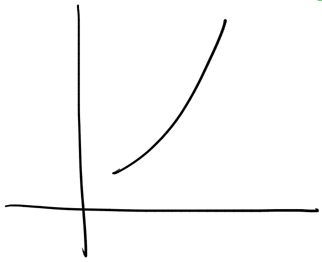
Quindi:

$$\min_D f = \min_{\mathring{D}} f = -\frac{1}{8} \quad \left(= f\left(\frac{1}{4},0\right)\right)$$

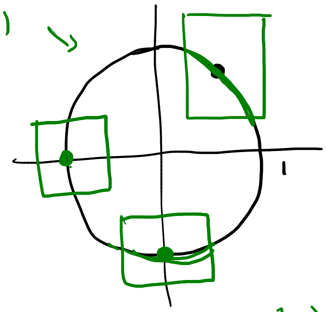
$$\max_D f = \max_{\partial D} f = 3 \quad (= f(-1,0))$$

Oss. preliminari su teor. funzione implicita



$\underbrace{x^2+y^2-1}_{g(x,y)} = 0$




$\nabla g(x,y) = (2x, 2y)$

Esempi (mostrano che la condizione $\nabla g \neq 0$ nel TFI è sufficiente ma non necessaria)

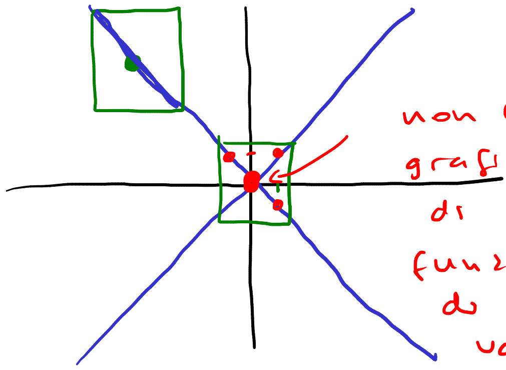
- $g(x,y) = x^2 - y^2 \qquad g \in C^1(\mathbb{R}^2, \mathbb{R})$

$\nabla g(x,y) = (2x, -2y) = \underbrace{(0,0)} \iff (x,y) = (0,0)$

$\underbrace{Z} = \{(x,y) \mid x^2 - y^2 = 0\}$

$x^2 = y^2$

$y = \pm x$



non è grafico di una funzione di una var.!

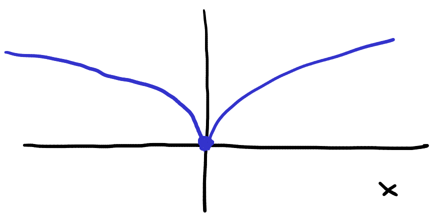
- $g(x,y) = x^2 - y^3 \qquad C^1$

$\nabla g(x,y) = (2x, -3y^2)$

$\nabla g(0,0) = (0,0)$

Però: $\underbrace{Z} = \{x^2 - y^3 = 0\}$

$y^3 = x^2 \qquad y = x^{2/3}$




grafico!

Esempio (gradiente ortogonale alle curve di livello)

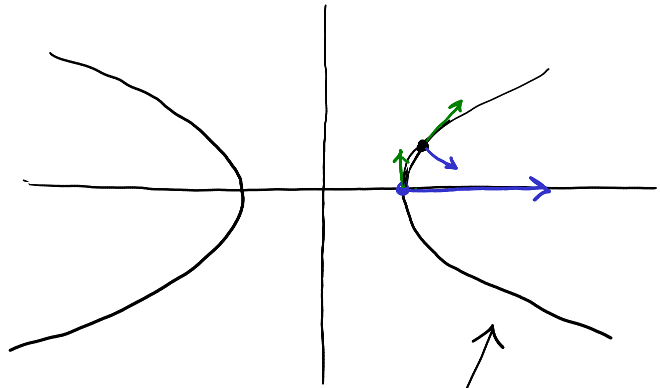
$g(x,y) = x^2 - y^2 \qquad c = 4$

$E_4 = \{(x,y) \mid x^2 - y^2 = 4\}$

$\uparrow$

$y^2 = x^2 - 4$

$y = \pm\sqrt{x^2-4}$



$\nabla g(x,y) = (2x, -2y)$

Determino la funzione h:

$x^2 - y^2 = 4 \qquad x^2 = 4 + y^2 \longrightarrow h(y) = \sqrt{4+y^2} \qquad y \in \mathbb{R}$

Parametrizzo: $r(y) = (\sqrt{4+y^2}, y)$

$$r'(y) = \left( \frac{y}{\sqrt{4+y^2}}, 1 \right)$$

In $(2,0)$ : • $\nabla g(2,0) = (4,0)$ , • $r'(0) = (0,1)$

In $(\sqrt{5},1)$ : • $\nabla g(\sqrt{5},1) = \left(\frac{2}{\sqrt{5}}, -2\right)$ • $r'(1) = \left(\frac{1}{\sqrt{5}}, 1\right)$

$$\left(\frac{2}{\sqrt{5}}, -2\right) \cdot \left(\frac{1}{\sqrt{5}}, 1\right) = 0$$

Esempio (applicazione del TML)

$f(x,y) = 2x^2 + y^2 - x \qquad C^1$

$Z = \{(x,y) \mid \underbrace{x^2+y^2-1}_{g(x,y)} = 0\} \qquad g \in C^1$

$\exists (x,y) \in Z$ t.c. $\nabla g(x,y) = (0,0)$ ?

$$\begin{cases} x^2+y^2-1=0 \\ 2x=0 \\ 2y=0 \end{cases} \qquad \begin{aligned} -1&=0 \; !! \\ x&=0 \\ y&=0 \end{aligned}$$

non ci sono punti singolari

Lagrangiana: $L(x,y,\lambda) = f(x,y) - \lambda\, g(x,y)$
$= 2x^2+y^2-x - \lambda(x^2+y^2-1)$

Punti staz:

$$\begin{cases} L_x(x,y,\lambda) = 4x-1-\lambda\cdot 2x = 0 \\ L_y(x,y,\lambda) = 2y-\lambda\cdot 2y = 0 \\ L_\lambda(x,y,\lambda) = -(x^2+y^2-1) = 0 \end{cases}$$

Candidati punti di estremo locale di $f$ su $Z$

$$\begin{cases} 4x-1-2\lambda x = 0 \\ 2y(1-\lambda) = 0 \\ x^2+y^2-1=0 \end{cases}$$

$\begin{cases} \dots \\ y=0 \\ x^2-1=0 \end{cases} \longrightarrow (\pm 1, 0)$

$\begin{cases} 4x-1-2x=0 \\ \lambda=1 \\ x^2+y^2-1=0 \end{cases}$ $\odot$

$\odot$ $\begin{cases} 2x-1=0 \\ x^2+y^2-1=0 \end{cases}$ $\qquad x=\frac{1}{2}, \quad y^2=\frac{3}{4}$ $\qquad \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Per concludere:

$f(1,0) = 1$

$f(-1,0) = 3 \leftarrow \max_Z f$

$f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{3}{4} \leftarrow \min_Z f$

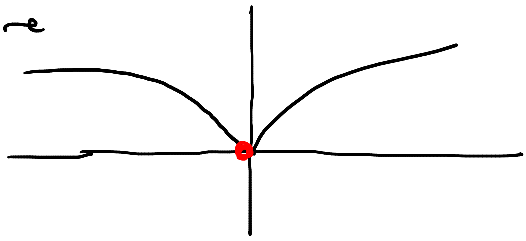
## Esempi (punti singolari)

- $\{(x,y) \mid \underbrace{x^2 - y^3}_{g(x,y)} = 0\} =: Z$

Punti singolari?

$$\begin{cases} x^2 - y^3 = 0 \\ 2x = 0 \\ -3y^2 = 0 \end{cases} \qquad \begin{matrix} 0 = 0 \\ x = 0 \\ y = 0 \end{matrix}$$

$\Rightarrow$ $(0,0)$ punto singolare

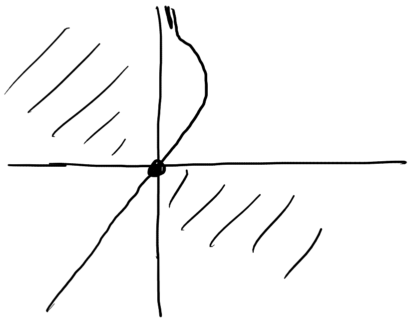


- $Z = \{(x,y) \mid \underbrace{x e^y - y}_{g(x,y)} = 0\}$

Punti sing.:

$$\begin{cases} x e^y - y = 0 \\ e^y = 0 \quad \longleftarrow \text{impossibile!} \\ x e^y - 1 = 0 \end{cases}$$

$x e^y - y = 0 \iff x = y e^{-y}$



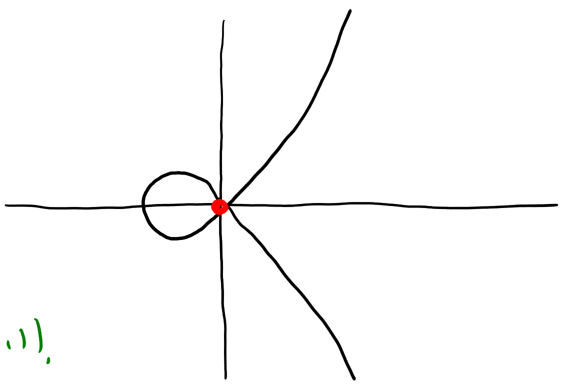
- $\{(x,y) \mid \overbrace{x^2(1+4x)}^{x^2+4x^3} - y^2 = 0\}$

$$\begin{cases} x^2(1+4x) - y^2 = 0 \\ 2x + 12x^2 = 0 \\ -2y = 0 \end{cases} \qquad \begin{cases} x^2(1+4x) = 0 \\ 2x(1+6x) = 0 \\ y = 0 \end{cases}$$

$(0,0)$ punto singol.

$y^2 = x^2(1+4x) \qquad 1+4x \geq 0$

$y = \pm\sqrt{x^2(1+4x)} = \pm |x| \sqrt{1+4x}$



Nota: è il sostegno di $r(t) = (t(t-1), t(t-1)(2t-1))$, già studiata