30 maggio 2025

Esempi (calcolo di aree)

- area della sup. sferica di raggio $r$

$\sigma(\varphi, \theta) = (r \sin\varphi \cos\theta,\ r\sin\varphi \sin\theta,\ r\cos\varphi)$

$(\varphi, \theta) \in [0, \pi] \times [0, 2\pi] =: K$

$\vdots$ (già calcolato)

$N_\sigma(\varphi, \theta) = r\sin\varphi\ \sigma(\varphi, \theta)$

$\Rightarrow \|N_\sigma(\varphi,\theta)\| = r\sin\varphi\ \|\sigma(\varphi,\theta)\| = r^2 \sin\varphi$

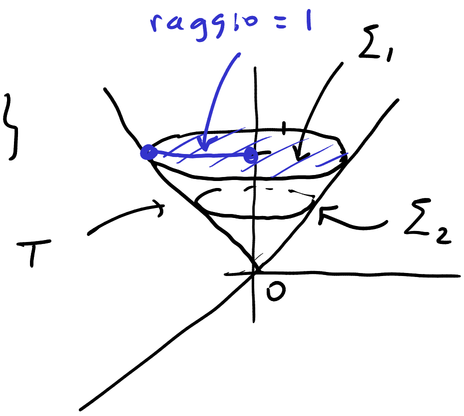
$$\text{area} = \iint_K \|N_\sigma(\varphi,\theta)\|\, d\varphi\, d\theta$$

$$= \iint_{[0,\pi]\times[0,2\pi]} r^2 \sin\varphi\, d\varphi\, d\theta$$

$$= r^2 \int_0^\pi \sin\varphi\, d\varphi \int_0^{2\pi} d\theta = r^2\, 2 \cdot 2\pi = 4\pi r^2$$

- area della frontiera di:

$T := \{(x,y,z) \mid \sqrt{x^2+y^2} \leq z \leq 1\}$




$\begin{cases} z = \sqrt{x^2+y^2} \\ z = 1 \end{cases}$

$$\underbrace{\text{area}(\partial T)}_{\pi(1+\sqrt{2})} = \underbrace{\text{area}(\Sigma_1)}_{\pi} + \underbrace{\text{area}(\Sigma_2)}_{\pi\sqrt{2}}$$

$\Sigma_2$ : $\sigma(u,v) = (u\cos v,\ u\sin v,\ u)$

$(u,v) \in [0,1] \times [0, 2\pi]$

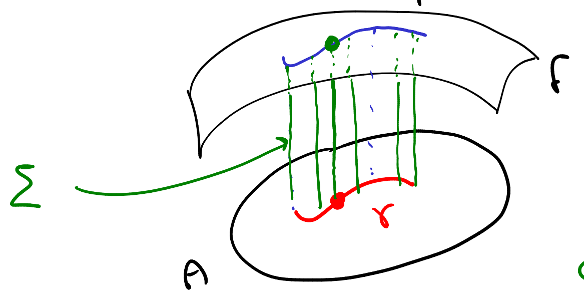
Già calcolato : $\|N_\sigma(u,v)\| = \sqrt{2}\, u$

$$\text{area}(\Sigma_2) = \iint_{[0,1]\times[0,2\pi]} \sqrt{2}\, u\, du = \sqrt{2}\int_0^1 u\, du \int_0^{2\pi} dv$$

$$= \sqrt{2} \cdot \frac{1}{2} \cdot 2\pi = \pi\sqrt{2}$$

## Interpretazione geometrica dell'integrale curvilineo

$\gamma : \quad r(t) = (x(t), y(t)) \qquad t \in [a,b]$




$$\Sigma : \quad \sigma(t, z) = (\underbrace{x(t), y(t), z}_{C^1})$$

$$K = \{ (t,z) \mid a \le t \le b, \; \underbrace{0}_{\alpha(t)} \le z \le \underbrace{f(r(t))}_{\beta(t)} \}$$

insieme normale rispetto all'asse $t$ nel piano $(t, z)$

$$\frac{\partial \sigma}{\partial t}(t,z) = (x'(t), y'(t), 0)$$

$$\frac{\partial \sigma}{\partial z}(t,z) = (0, 0, 1)$$

$$N_\sigma(t,z) = (y'(t), -x'(t), 0)$$

$$\| N_\sigma(t,z) \| = \sqrt{y'(t)^2 + x'(t)^2 + 0^2} = \| r'(t) \| \neq 0$$

$\gamma$ regolare

$\Rightarrow \sigma$ è param. regolare

$$\text{area}(\Sigma) = \iint_K \| N_\sigma(t,z) \| \, dt \, dz = \iint_K \| r'(t) \| \, dt \, dz$$

$$= \int_a^b \left( \int_0^{f(r(t))} \| r'(t) \| \, dz \right) dt$$

$$= \int_a^b \| r'(t) \| \, ( f(r(t)) - 0 ) \, dt$$

$$= \int_a^b f(r(t)) \, \| r'(t) \| \, dt \overset{\text{def}}{=} \int_\gamma f \, ds$$

## Esempi (calcolo di flussi)

- Flusso di: $F(x,y,z) = (0,0,z)$ $\quad F \in C(\mathbb{R}^3, \mathbb{R}^3)$

sup. $\sigma(u,v) = (\overset{x}{u\cos v}, \overset{y}{u\sin v}, \overset{z}{v})$ $\quad (u,v) \in [0,1] \times [0,2\pi]$

Già visto: • $\sigma$ sup. regolare con bordo, quindi è orientabile

• $N_\sigma(u,v) = (\sin v, -\cos v, u)$

$$\Phi_\Sigma(F) = \iint_{[0,1]\times[0,2\pi]} (0,0,v) \cdot (\sin v, -\cos v, u)\, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} uv \, du\, dv = \int_0^1 u\, du \cdot \int_0^{2\pi} v\, dv = \dots$$

- flusso di $F(x,y,z) = (x,y,0)$ <u>USCENTE</u>

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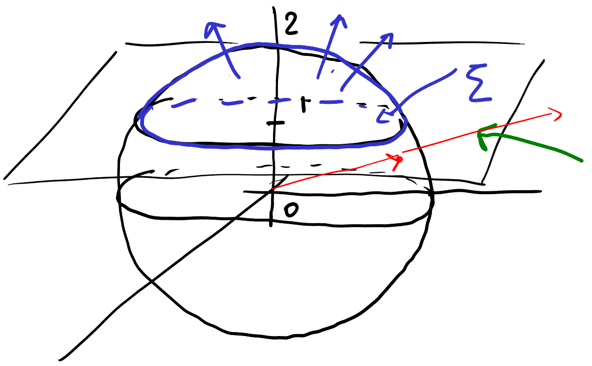



$z = 1 \iff 2\cos\varphi = 1$
$\iff \cos\varphi = \frac{1}{2} \iff \varphi = \frac{\pi}{3}$

$\Sigma : \sigma(\varphi,\theta) = (\overset{x}{2\sin\varphi\cos\theta}, \overset{y}{2\sin\varphi\sin\theta}, \overset{z}{2\cos\varphi})$

$(\varphi,\theta) \in \left[0, \frac{\pi}{3}\right] \times [0, 2\pi]$

Già noto: $N_\sigma(\varphi,\theta) = 2\sin\varphi \, \sigma(\varphi,\theta)$

$= (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, \underbrace{4\sin\varphi\cos\varphi}_{\geq 0})$

$0 \leq \varphi \leq \frac{\pi}{2} \Rightarrow N_\sigma$ punta verso l'alto

$\Phi_\Sigma(F)$

$$= \iint_{[0,\frac{\pi}{3}]\times[0,2\pi]} (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 0)\cdot(4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

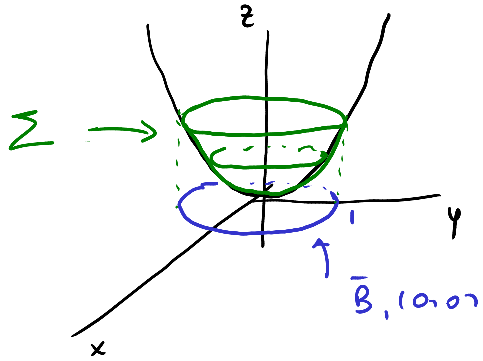
$$= \iint_{[0,\frac{\pi}{3}]\times[0,2\pi]} (8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 0)\, d\varphi\, d\theta$$

$$= \iint_{[0,\frac{\pi}{3}]\times[0,2\pi]} 8\sin^3\varphi\, d\varphi\, d\theta = 8\int_0^{\frac{\pi}{3}} (1-\cos^2\varphi)\sin\varphi\, d\varphi \cdot 2\pi$$

$$= \dots$$

- $F(x,y,z) = (y, x, z)$

flusso diretto verso il basso attraverso sup. grafico associata a $f(x,y) = x^2+y^2$, $(x,y)\in \bar{B}_1(0,0)$




$\sigma(u,v) = (u, v, u^2+v^2)$

$(u,v)\in \bar{B}_1(0,0)$

( sup. grafico è reg. con bordo $\Rightarrow$ orientabile )

È noto che: $N_\sigma(u,v) = (-2u, -2v, \underbrace{1}_{>0})$ punta verso l'alto!

$$\Phi_\Sigma(F) = -\iint_{\bar{B}_1(0,0)} (v, u, u^2+v^2)\cdot(-2u, -2v, 1)\, du\, dv$$

$$= -\iint_{\bar{B}_1(0,0)} (-2uv - 2uv + u^2+v^2)\, du\, dv$$

$$= \iint_{\bar{B}_1(0,0)} (4uv - (u^2+v^2))\, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} (4\rho^2\cos\theta\sin\theta - \rho^2)\,\rho\, d\rho\, d\theta$$

$$= \int_0^1 4\rho^3 d\rho \underbrace{\int_0^{2\pi} \cos\theta \sin\theta \, d\theta}_{=0} - \int_0^1 \rho^3 d\rho \cdot 2\pi = -\frac{\pi}{2}$$

## Esempi (calcolo di rotore)

- $F(x,y,z) = (xy, x^2, yz)$ $\qquad F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$

$$\operatorname{rot} F(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & yz \end{vmatrix} = (z-0,\ 0-0,\ 2x-x)$$

$$= (z, 0, x)$$

- $F(x,y,z) = (x,y,z)$

$$\operatorname{rot} F(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = (0-0,\ 0-0,\ 0-0)$$

$$= (0,0,0)$$

- $F(x,y,z) = (y, -x, 0)$

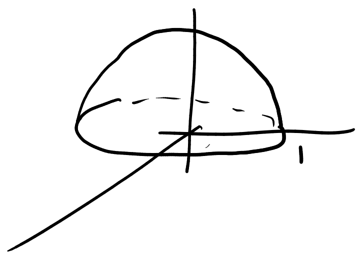
$$\operatorname{rot} F(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = (0, 0, -1-1)$$

$$= (0, 0, -2)$$

## Esempio (sul teorema di Stokes)

$F(x,y,z) = (xy, x^2, yz)$

$\Sigma$ :



Già calcolato:

$\operatorname{rot} F(x,y,z) = (z, 0, x)$

$\sigma(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

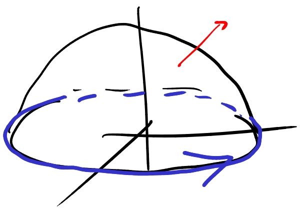
$(\varphi,\theta) \in \underbrace{[0, \pi/2] \times [0, 2\pi]}_{K}$

$$\Phi_\Sigma(\operatorname{rot} F) = \int_\Sigma \operatorname{rot} F \cdot n \, dS$$

$$= \iint_K (\cos\varphi, 0, \sin\varphi\cos\theta) \cdot (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi) \, d\varphi \, d\theta$$

$$= \iint_K (\cos\varphi \sin^2\varphi \cos\theta + \sin^2\varphi \cos\varphi \cos\theta) \, d\varphi \, d\theta$$

$$= 2 \iint_{[0,\pi/2]\times[0,2\pi]} \cos\varphi \sin^2\varphi \cos\theta \, d\theta = \dots = \underline{0}$$



$$\partial\Sigma^+ : \quad r(t) = (\cos t, \sin t, 0) \qquad t \in [0, 2\pi]$$

$$\underline{\int_{\partial\Sigma^+} F(P) \cdot dP} = \int_0^{2\pi} (\cos t \sin t, \cos^2 t, 0) \cdot (-\sin t, \cos t, 0) \, dt$$

$$= \int_0^{2\pi} (-\cos t \sin^2 t + \cos^3 t) \, dt = \dots = \underline{0}.$$