29 maggio 2025

Verifico che se $F \in C^1(A, \mathbb{R}^n)$, allora:

F conservativo $\Rightarrow$ F chiuso

F conservativo $\Leftrightarrow$ $\exists f \in C^1(A, \mathbb{R})$ t.c. $\nabla f = F$

$$\Updownarrow$$

$$\forall i \in \{1, \dots, n\} : \quad \frac{\partial f}{\partial x_i} = \underbrace{F_i}_{\in C^1}$$

Oss: $f \in C^2(A, \mathbb{R})$

Fisso $i, j \in \{1, \dots, n\}$, $i \neq j$:

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i} \overset{\text{teor. di Schwarz}}{=} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial F_j}{\partial x_i} \qquad \square$$

Esempio (che mostra che chiuso $\not\Rightarrow$ conservativo)

$$\underset{\uparrow \in C^1}{F(x,y)} = \Big(\overbrace{\underbrace{-\frac{y}{x^2+y^2}}_{\in C^1}}^{F_1}, \overbrace{\underbrace{\frac{x}{x^2+y^2}}_{\in C^1}}^{F_2} \Big) \qquad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

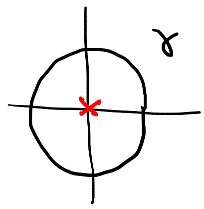
$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$\frac{\partial F_1}{\partial y}(x,y) = -\frac{x^2+y^2 - y \cdot 2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x}(x,y) = \frac{x^2+y^2 - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad //$$

$\Rightarrow F$ è chiuso

Considero $r(t) = (\underbrace{\cos t}_{x}, \underbrace{\sin t}_{y})$ $\quad t \in [0, 2\pi]$



Calcolo

$$\int_\gamma F(P)\cdot dP = \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$$

$$= \int_0^{2\pi} \left(-\frac{\sin t}{\cos^2 t+\sin^2 t}, \frac{\cos t}{\cos^2 t+\sin^2 t}\right)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi \neq 0$$

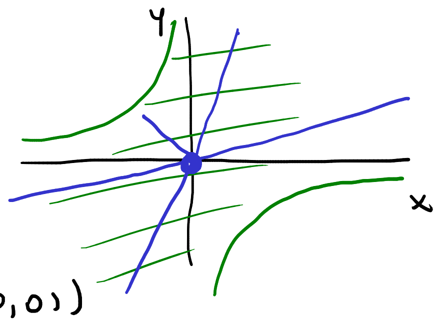
$\Rightarrow$ F non è conservativo in $\mathbb{R}^2 \setminus \{(0,0)\}$

Esempi (sul teor. di Poincaré)

- $$F(x,y,z) = \Big(\overbrace{\frac{y}{1+xy}+z}^{F_1}, \overbrace{\frac{x}{1+xy}}^{F_2}, \overbrace{x}^{F_3}\Big) \qquad F \in C^1(A, \mathbb{R}^3)$$

$$A = \{(x,y,z) \mid 1+xy > 0\}$$

$xy > -1$



Oss: A è aperto e stellato (rispetto a $(0,0,0)$)

Per il teor. di Poincaré: F cons. in A $\Leftrightarrow$ F chiuso in A

$$\frac{\partial F_1}{\partial y}(x,y,z) = \frac{1+xy-yx}{(1+xy)^2} = \frac{1}{(1+xy)^2}$$

$\quad$ " $\quad$ ✓

$$\frac{\partial F_2}{\partial x}(x,y,z) = \frac{1+xy-xy}{(1+xy)^2} = \frac{1}{(1+xy)^2}$$

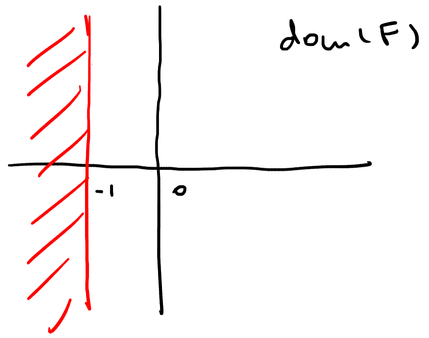
$\frac{\partial F_1}{\partial z}(x,y,z) = 1$ $\qquad$ $\frac{\partial F_2}{\partial z}(x,y,z) = 0$

" ✓ $\qquad$ " ✓

$\frac{\partial F_3}{\partial x}(x,y,z) = 1$ $\qquad$ $\frac{\partial F_3}{\partial y}(x,y,z) = 0$

Quindi: F è chiuso, pertanto conservativo in A.

- $\vec{F}(x,y) = \left(\overbrace{\frac{1+y}{1+x}}^{F_1(x,y)} , \overbrace{\ln(1+x)}^{F_2(x,y)} \right)$



$dom(F) = \{(x,y) \mid 1+x>0\}$

↑

aperto, convesso

($\Rightarrow$ stellato)

$F_1(x,y) = \frac{1}{1+x}(1+y)$

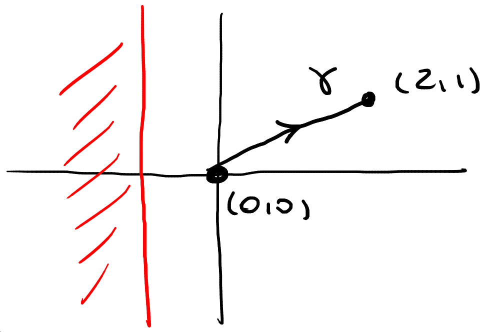
F di classe C^1

Per teor. di Poincaré: F cons. $\Leftrightarrow$ F chiuso

$\forall (x,y) \in dom(F)$:
$$\frac{\partial F_1}{\partial y}(x,y) = \frac{1}{1+x} \cdot 1 \quad ✓$$
$$\|$$
$$\frac{\partial F_2}{\partial x}(x,y) = \frac{1}{1+x} \cdot 1$$

Quindi: F è conservativo (nel proprio ins. di definizione)

Calcolo l'integrale di F sul segmento congiungente $(0,0)$ e $(2,1)$



Procedo in tre modi.

① Parametrizzo γ

$$r(t) = (0,0) + t((2,1)-(0,0)) \qquad t \in [0,1]$$
$$= (\underbrace{2t}_{x}, \underbrace{t}_{y})$$

$$r'(t) = (2,1)$$

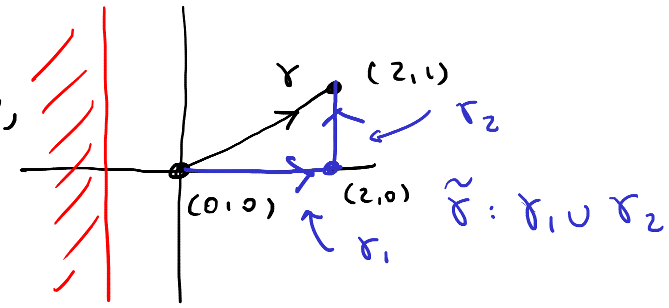
$$\int_\gamma F(P) \cdot dP = \int_0^1 F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 \left(\frac{1+t}{1+2t}, \ln(1+2t) \right) \cdot (2,1)\, dt$$

$$= \int_0^1 \left(\frac{1+t}{1+2t} \cdot 2 + \ln(1+2t) \right) dt$$

$$= \int_0^1 \Big(\underbrace{\frac{1}{1+2t}}_{\ln \dots} + \underbrace{1}_{} + \underbrace{\ln(1+2t)}_{\text{per parti}} \Big)\, dt = \dots$$

(2) Siccome F è conservativo,



$$\int_\gamma F(P)\cdot dP = \int_{\tilde\gamma} F(P)\cdot dP =$$

γ e $\tilde\gamma$ hanno gli stessi estremi

$\gamma_1: \ r(t) = (t, 0) \quad t \in [0,2]$

$\gamma_2: \ r(t) = (2, t) \quad t \in [0,1]$

$$= \int_{\gamma_1} F(P)\cdot dP + \int_{\gamma_2} F(P)\cdot dP$$

$$= \int_0^2 \Big(\frac{1+0}{1+t}\cdot 1 + \ln(1+t)\cdot 0 \Big)\, dt + \int_0^1 \Big(\frac{1+t}{1+2}\cdot 0 + \ln(1+2)\cdot 1 \Big)\, dt$$

$$= \int_0^2 \frac{1}{1+t}\, dt + \int_0^1 \ln 3 \, dt = \ln 3 + \ln 3 = 2 \ln 3$$

(3) Determino un potenziale di $\overline{F}$ e applico TFCI.

$f : \mathrm{dom}(F) \to \mathbb{R}$ t.c. $\forall (x,y) \in \mathrm{dom}(F)$:

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = \dfrac{1+y}{1+x} & (a) \\[2ex] \dfrac{\partial f}{\partial y}(x,y) = \ln(1+x) & (b) \end{cases}$$

Integro (a) rispetto a x:

$$f(x,y) = (1+y)\ln(1+x) + g(y)$$

Sostituisco in (b):

$$1 \cdot \ln(1+x) + g'(y) = \ln(1+x)$$

$\iff \ g'(y) = 0$; scelgo $g(y) = 0 \ \ \forall y$

Quindi ottengo la primitiva

$$f(x,y) = (1+y)\ln(1+x)$$

Applico FFCI:

$$\int_\gamma F(P)\cdot dP = f(2,1) - f(0,0) = 2\ln 3 - 0 = 2\ln 3.$$

- $$F(x,y) = \left(\overbrace{\frac{x-y}{x^2+y^2}}^{F_1(x,y)} , \overbrace{\frac{x+y}{x^2+y^2}}^{F_2(x,y)} \right)$$

$\underbrace{dom(F) = \mathbb{R}^2 \setminus \{(0,0)\}}_{\text{aperto, non stellato}}$, F di classe C^1

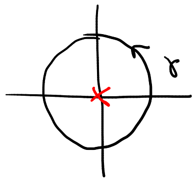
Verifico se F è chiuso:

$$\frac{\partial F_1}{\partial y}(x,y) = \frac{-(x^2+y^2) - (x-y)\,2y}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2y^2}{(\;..\;)^2}$$

$$\frac{\partial F_2}{\partial x}(x,y) = \frac{x^2+y^2-(x+y)\,2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2-2xy}{(\;...\;)^2}$$

Quindi: F è chiuso.

Non posso applicare il teor. di Poincaré (perché $dom(F)$ non è stellato); calcolo una circuitazione di F



Scelgo $\quad r(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$$\int_\gamma F(P)\cdot dP = \int_0^{2\pi} \left(\frac{\cos t - \sin t}{1} , \frac{\cos t + \sin t}{1} \right)\cdot(-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} (-\cos t \cancel{\sin t} + \sin^2 t + \cos^2 t + \cancel{\sin t} \cos t)\, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi \neq 0$$

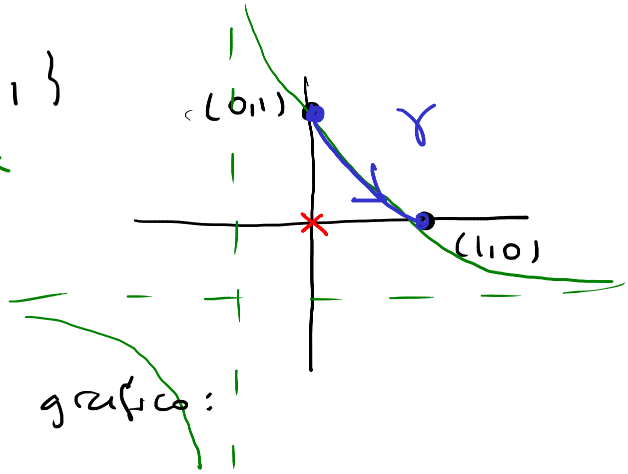
Quindi: F <u>non</u> è conservativo in $\mathrm{dom}(F)$

Calcolo l'int. di F sulla curva semplice di estremi $(0,1)$ e $(1,0)$ e sostegno contenuto in

$$\{(x,y) \mid xy + x + y = 1\}$$

$$(x+1)y = 1-x$$

$$y = \frac{1-x}{1+x}$$



Parametrizzo γ come curva grafico:

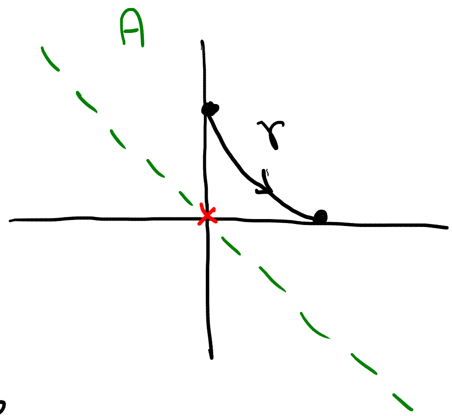
$$r(t) = \left(t, \frac{1-t}{1+t}\right), \quad t \in [0,1]$$

$$\int_\gamma F(P) \cdot dP = \int_0^1 \left(\frac{t - \left(\frac{1-t}{1+t}\right)}{t^2 + \left(\frac{1-t}{1+t}\right)^2}, \frac{t + \left(\frac{1-t}{1+t}\right)}{t^2 + \left(\frac{1-t}{1+t}\right)^2} \right) \cdot \left(1, \frac{-(1+t)-(1-t)}{(1+t)^2}\right) dt$$

$$= \int_0^1 \left(\frac{t - \frac{1-t}{1+t}}{t^2 + \left(\frac{1-t}{1+t}\right)^2} + \frac{t + \frac{1-t}{1+t}}{t^2 + \left(\frac{1-t}{1+t}\right)^2} \cdot \frac{-2}{(1+t)^2} \right) dt$$

HELP!!!

Provo una strada alternativa.



$A := \{(x,y) \mid x + y > 0\}$

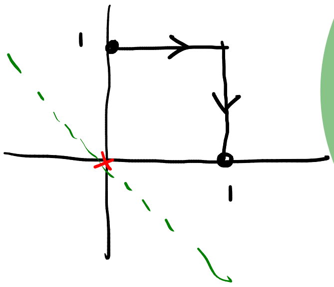
- $\gamma \subset A$
- A ap., convesso $\Rightarrow$ stellato

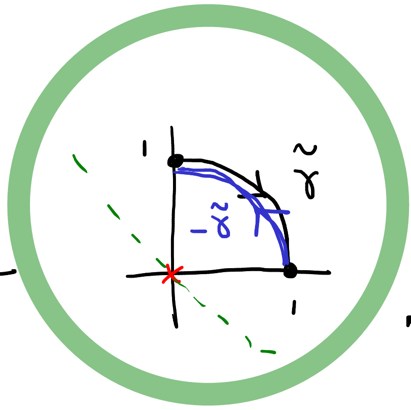
• Ristretto ad A, F è chiuso

Per il teor. di Poincaré : $F_{|A}$ è conservativo.

$$\int_\gamma F(P)\cdot dP = \int_\gamma F_{|A}(P)\cdot dP = \int_{\tilde\gamma} F_{|A}(P)\cdot dP$$

$\tilde\gamma$ qualsiasi "percorso" di estremi $(0,1)$ e $(1,0)$ CONTENUTO in A





$r(t) = (\cos t, \sin t)$ $\quad t \in [0, \frac{\pi}{2}]$

$$\int_{-\tilde\gamma} F(P)\cdot dP = \int_0^{\frac{\pi}{2}} \underbrace{F(r(t))\cdot r'(t)}_{=1 \text{ (già calcolato)}}\, dt = \frac{\pi}{2}$$

$$\Rightarrow \int_{\tilde\gamma} F(P)\cdot dP = -\frac{\pi}{2} \qquad \square$$

Esempi (integrali di superficie)

• $f(x,y,z) = z$ $\quad$ dom$(f) = \mathbb{R}^3$, $\quad f$ continua

$\sigma(u,v) = (\underbrace{u\cos v}, \underbrace{u\sin v}, \underbrace{u})$ $\quad (u,v) \in \underbrace{[0,1]\times[0,2\pi]}_{K}$

di classe C^1

"iniettività" ✓

$$\frac{\partial\sigma}{\partial u}(u,v) = (\cos v, \sin v, 1)$$

$e_1 \quad e_2 \quad e_3$

$$\frac{\partial\sigma}{\partial v}(u,v) = (-u\sin v, u\cos v, 0)$$

$$N_\sigma(u,v) = (-u\cos v, -u\sin v, u)$$

$$\| N_\sigma(u,v) \| = \sqrt{u^2\cos^2 v + u^2 \sin^2 v + u^2} = \sqrt{2}\, u$$

Oss: $(u,v) \in \text{int}(K) \Rightarrow u \neq 0 \Rightarrow N_\sigma(u,v) \neq 0$

$\Rightarrow$ la sup. è regolare

$$\int_\Sigma f \, dS = \iint_K f(\sigma(u,v)) \, \| N_\sigma(u,v) \| \, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} u \cdot \sqrt{2} u \, du\, dv = \sqrt{2} \int_0^1 u^2 du \cdot \int_0^{2\pi} dv$$

$$= \frac{2\sqrt{2}\pi}{3}$$

D: che sup. stiamo parlando ?

$$\sigma(u,v) = (\overset{x}{\overbrace{u\cos v}}, \overset{y}{\overbrace{u\sin v}}, \overset{z}{\overbrace{u}})$$ SUP. CONICA

$$x^2 + y^2 = z^2$$

$$z = \pm \sqrt{x^2+y^2} \qquad\qquad z = \sqrt{x^2+y^2}$$

$z = u \in [0,1]$

- $f(x,y,z) = z$ come prima

$\sigma(u,v) = (u\cos v, u \sin v, v)$ $\quad (u,v) \in \underbrace{[0,1]\times[0,2\pi]}_{K}$

$\in C^1$

σ è iniettiva in tutto K

Calcolo

$$\frac{\partial \sigma}{\partial u}(u,v) = (\cos v, \sin v, 0)$$

$$\frac{\partial \sigma}{\partial v}(u,v) = (-u\sin v, u\cos v, 1)$$

$$N_\sigma(u,v) = (\sin v, -\cos v, u)$$

$$\|N_\sigma(u,v)\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2} \neq 0$$

$$\forall (u,v) \in K$$

$$\int_\Sigma f\, dS = \iint_K f(\sigma(u,v))\, \|N_\sigma(u,v)\|\, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} v \sqrt{1+u^2}\, du\, dv = \int_0^1 \sqrt{1+u^2}\, du \cdot \int_0^{2\pi} v\, dv$$

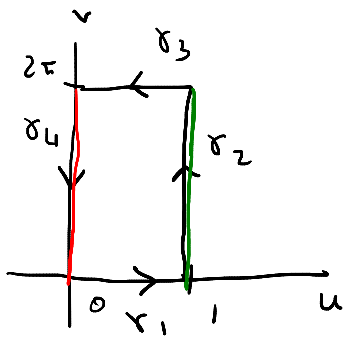
$\sqrt{1+u^2} = u+s$
...

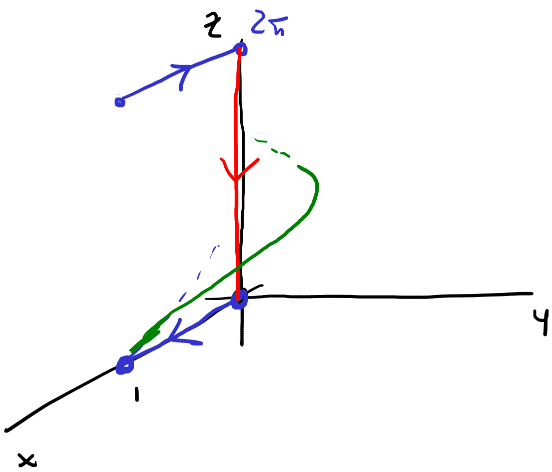
$= \ldots$

Oss: K dom. regolare in $\mathbb{R}^2$

$\Rightarrow$ (Σ, σ) è sup. regolare <u>con bordo</u>

Determino il bordo:





γ_1 : $r(t) = (t,0)$, $t \in [0,1] \rightsquigarrow (\sigma \circ r)(t) = \sigma(t,0) = (t,0,0)$

γ_2 : $r(t) = (1, t)$ $t \in [0, 2\pi] \rightsquigarrow \sigma(1,t) = (\cos t, \sin t, t)$

γ_3 : $r(t) = (1-t, 2\pi)$ $t \in [0,1] \rightsquigarrow \sigma(1-t, 2\pi)$

$= (1-t, 0, 2\pi)$

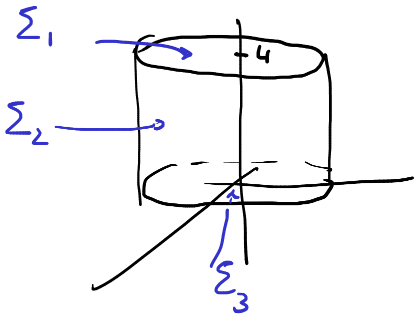
γ_4 : $r(t) = (0, 2\pi - t)$ $t \in [0, 2\pi] \rightsquigarrow \sigma(0, 2\pi - t) = (0, 0, 2\pi - t)$

Esempio (integrale su superficie regolare a pezzi)

$f(x,y,z) = z$

Σ = frontiera di $\{ (x,y,z) \mid x^2+y^2 \leq 9, \ 0 \leq z \leq 4 \}$

$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$



$$\int_\Sigma f\,dS = \int_{\Sigma_1} f\,dS + \int_{\Sigma_2} f\,dS + \int_{\Sigma_3} f\,dS$$

Σ_1 : $\sigma(u,v) = (u, v, 4) \qquad (u,v) \in \bar{B}_3(0,0)$

$\sigma(u,v) = (u, v, f(u,v))$

$N_\sigma(u,v) = (0,0,1) \qquad N_\sigma(u,v) = (-f_u(u,v), -f_v(u,v), 1)$

$$\Rightarrow \int_{\Sigma_1} f\,dS = \iint_{\bar{B}_3(0,0)} 4 \cdot 1 \, du\,dv = 4\pi \cdot 9 = 36\pi$$

Σ_3 : $\sigma(u,v) = (u,v,0) \qquad (u,v) \in \bar{B}_3(0,0)$

$N_\sigma(u,v) = (0,0,1)$

$$\int_{\Sigma_3} f\,dS = \iint_{\bar{B}_3(0,0)} 0 \cdot 1 \, du\,dv = 0$$

Σ_2 : $\sigma(\theta, z) = (3\cos\theta, 3\sin\theta, z) \qquad (\theta, z) \in [0,2\pi] \times [0,4]$

$\vdots$ (già calcolato)

$N_\sigma(\theta, z) = (3\cos\theta, 3\sin\theta, 0)$

$\| N_\sigma(\theta,z) \| = 3$

$$\int_{\Sigma_2} f\,dS = \iint_{[0,2\pi]\times[0,4]} z \cdot 3 \, d\theta\,dz = 3 \cdot 2\pi \cdot 8 = 48\pi$$

Conclusione: $\displaystyle\int_\Sigma f\,dS = 36\pi + 48\pi + 0 = 84\pi$.