27 maggio 2025

Esempi (calcolo di integrali)

- $F(x,y) = (y, xy)$ F continuo, $\mathrm{dom}(F) = \mathbb{R}^2 \Rightarrow \gamma \subset \mathrm{dom}(F)$

$r(t) = (\cos t, \sin t)$ ($x = \cos t$, $y = \sin t$), $t \in [0, \frac{\pi}{2}]$

$\forall t: \quad F(r(t)) = F(\cos t, \sin t) = (\sin t, \cos t \sin t)$

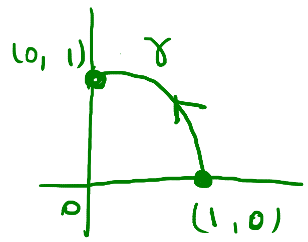
$r'(t) = (-\sin t, \cos t)$

$\Rightarrow \forall t: \quad F(r(t)) \cdot r'(t) = (\sin t, \cos t \sin t) \cdot (-\sin t, \cos t)$

$= -\sin^2 t + \cos^2 t \sin t$

Quindi:

$$\int_\gamma F(P) \cdot dP = \int_0^{\pi/2} F(r(t)) \cdot r'(t)\, dt$$



$$= \int_0^{\pi/2} (-\underbrace{\sin^2 t}_{= \frac{1-\cos 2t}{2}} + \cos^2 t \sin t)\, dt$$

$$= \left[-\frac{t}{2} + \frac{\sin 2t}{4} - \frac{1}{3}\cos^3 t \right]_0^{\pi/2} = -\frac{\pi}{4} + \frac{1}{3} \neq -\frac{1}{3}$$

- $F(x,y) = (y, xy)$ $\mathrm{dom}(F) = \mathbb{R}^2$, F cont. $\Rightarrow \gamma \subset \mathrm{dom}(F)$ ✓

curva grafico associata a $f(x) = 1 - x$, $x \in [0,1]$

$r(t) = (t, 1-t)$ ($x = t$, $y = 1-t$), $t \in [0,1]$

$\forall t: \quad F(r(t)) = F(t, 1-t) = (1-t, t(1-t))$

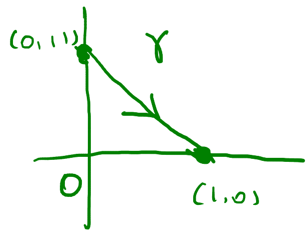
$r'(t) = (1, -1)$

$\Rightarrow F(r(t)) \cdot r'(t) = 1 - t - t(1-t) = (1-t)^2$

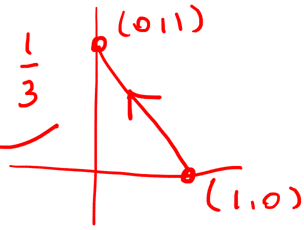
Quindi:

$$\int_\gamma F(P)\cdot dP = \int_0^1 F(r(t))\cdot r'(t)\,dt = \int_0^1 (1-t)^2\,dt$$

$$= \left[-\frac{(1-t)^3}{3} \right]_0^1 = \frac{1}{3}$$



$$\int_\gamma F(P)\cdot dP = -\frac{1}{3}$$

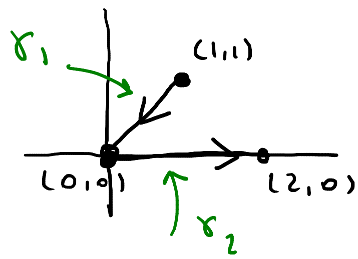


Esempi

- $F(x,y) = (e^{x+y}, x^2)$ $\quad F \in C^1(\mathbb{R}^2,\mathbb{R}^2)$ ✓

  γ: poligonale di vertici $(1,1)$, $(0,0)$, $(2,0)$

$$\int_\gamma F(P)\cdot dP = \int_{\gamma_1} F(P)\cdot dP + \int_{\gamma_2} F(P)\cdot dP$$

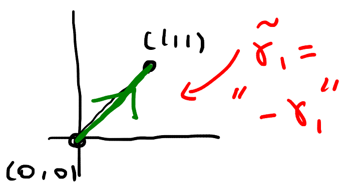


γ_1: $\quad r(t) = (t,t) \qquad t \in [0,1]$

$\qquad r'(t) = (1,1)$

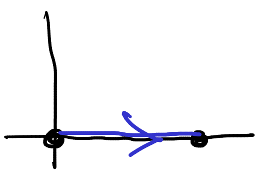
$$-\int_{\gamma_1} F(P)\cdot dP = \int_0^1 F(r(t))\cdot r'(t)\,dt = \int_0^1 (e^{t+t}, t^2)\cdot(1,1)\,dt$$

$$= \int_0^1 (e^{2t} + t^2)\,dt = \frac{e^2-1}{2} + \frac{1}{3}$$



γ_2: $\quad r(t) = (t,0) \qquad t \in [0,2]$

$\qquad r'(t) = (1,0)$



✓

$$\int_{\gamma_2} F(P)\cdot dP = \int_0^2 F(t,0)\cdot(1,0)\,dt = \int_0^2 (e^{t+0}, t^2)\cdot(1,0)\,dt$$

$$= \int_0^2 e^t\,dt = e^2 - 1$$

Conclusione: $\int_\gamma F(P)\cdot dP = -\left(\frac{e^2-1}{2}+\frac{1}{3}\right)+e^2-1 = \frac{e^2-1}{2}-\frac{1}{3}$

- $F(x,y) = (y^2, x^3)$ $\quad F\in C^1(\mathbb{R}^2,\mathbb{R}^2)$ ✓

γ: circonf. di centro $(0,0)$, raggio 2
percorsa in senso antiorario

$r(t) = (2\cos t, 2\sin t) \quad t\in[0,2\pi]$

$r'(t) = (-2\sin t, 2\cos t)$

$$\int_\gamma F(P)\cdot dP = \int_0^{2\pi} (4\sin^2 t, 8\cos^3 t)\cdot(-2\sin t, 2\cos t)\, dt$$

$$= \int_0^{2\pi} (-8\sin^3 t + 16\cos^4 t)\, dt = \dots$$

$\sin^3 t = (1-\cos^2 t)\sin t$ sostituz.

$\cos^4 t = \left(\frac{1+\cos 2t}{2}\right)^2$

$\cos^2 2t = \frac{1+\cos 4t}{2} \dots$

Esempio (come determinare un potenziale)

Determinare un potenziale di:

$$F(x,y,z) = \left(e^y + 2xz,\ xe^y - \frac{1}{y-2},\ x^2+z\right)$$

$\mathrm{dom}(F) = \{(x,y,z) \mid y \neq 2\}$ aperto, non connesso
$=: A$

$= A_+ \cup A_-$

$A_+ = \{(x,y,z) \mid y > 2\}$, $A_- = \{(x,y,z) \mid y < 2\}$

aperti connessi (perché convessi)

$f: A \to \mathbb{R}$ potenziale di F $\iff \forall (x,y,z)$:

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = e^y + 2xz & (1) \\ \frac{\partial f}{\partial y}(x,y,z) = xe^y - \frac{1}{y-2} & (2) \\ \frac{\partial f}{\partial z}(x,y,z) = x^2 + z & (3) \end{cases}$$

Parto da (1) e integro rispetto a x:

$$f(x,y,z) = xe^y + x^2 z + g(y,z) \qquad (*)$$

"costante additiva" (rispetto a x)

Sostituisco l'espressione trovata in (2):

$$\cancel{xe^y} + 0 + \frac{\partial g}{\partial y}(y,z) = \cancel{xe^y} - \frac{1}{y-2}$$

$$\Rightarrow \frac{\partial g}{\partial y}(y,z) = -\frac{1}{y-2}$$

Integro rispetto a y:

"costante additiva" rispetto a y

$$g(y,z) = -\ln|y-2| + h(z)$$

Sostituisco in $(*)$:

$$f(x,y,z) = xe^y + x^2 z - \ln|y-2| + h(z) \qquad (**)$$

Sostituisco in (3):

$$0 + \cancel{x^2} - 0 + h'(z) = \cancel{x^2} + z$$

$$\Rightarrow h'(z) = z$$

Scelgo $h(z) = \frac{z^2}{2}$. Sostituisco in $(**)$:

$$f(x,y,z) = xe^y + x^2 z - \ln|y-2| + \frac{z^2}{2} \qquad (x,y,z) \in A$$

□

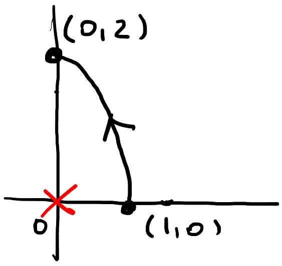
Esempi (applicazione della FFCI)

① $F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$ $\quad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

- $r(t) = (\cos t, 2\sin t)$ $\quad t \in [0, \frac{\pi}{2}]$

Già noto: F è conservativo, con potenziale

$$f(x,y) = \frac{1}{2} \ln(x^2+y^2)$$



OSS: $\gamma \subset \text{dom}(F)$

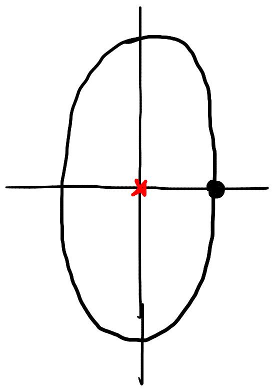
FFCI:

$$\int_\gamma F(P) \cdot dP = f\left(r\left(\frac{\pi}{2}\right)\right) - f\left(r(0)\right)$$

$$= f(0,2) - f(1,0) = \frac{1}{2}\ln 4 - \frac{1}{2}\ln 1$$

$$= \ln 2 .$$

- $r(t) = (\cos t, 2\sin t)$ $\quad t \in [0, 2\pi]$



curva chiusa

$$\Rightarrow \int_\gamma F(P) \cdot dP = 0$$

② $F(x,y,z) = \left(e^y + 2xz,\ x e^y - \frac{1}{y-2},\ x^2 + z \right)$

Già visto: $\text{dom}(F) = \mathbb{R}^3 \setminus \{ y = 2 \}$

Potenziale:

$$f(x,y,z) = x e^y + x^2 z - \ln|y-2| + \frac{z^2}{2}$$

- $r(t) = (\cos t, \sin t, t)$ $\quad t \in [0, 2\pi]$ $\quad$ elica!

$\sin t \neq 2$

$\Rightarrow \quad \gamma \subset \mathrm{dom}(F)$

FFCI: $\displaystyle\int_\gamma F(P)\cdot dP = f(r(2\pi)) - f(r(0))$

$= f(1,0,2\pi) - f(1,0,0)$

$= \cancel{1} + 2\pi - \cancel{\ln 2} + 2\pi^2 - \cancel{1} + \cancel{\ln 2}$

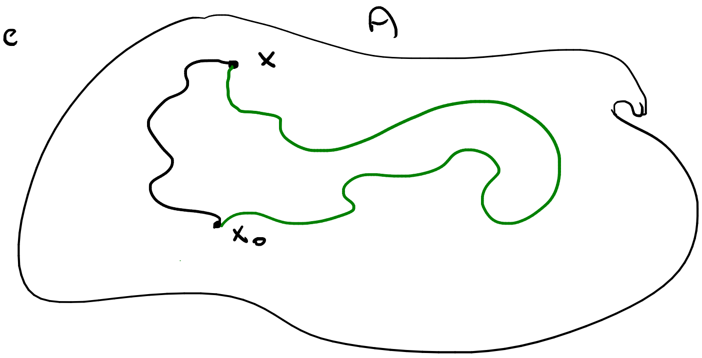
$= 2\pi + 2\pi^2 .$

- $r(t) = (\cos t, \sin t, \cos^3 t) \qquad t \in [0, 2\pi]$

  $\sin t \neq 2 \quad \Rightarrow \quad \gamma \subset \mathrm{dom}(F)$

  $r(0) = r(2\pi) \quad \Rightarrow \quad \displaystyle\int_\gamma F(P)\cdot dP = 0 .$

Cenno sulla dimostrazione di "b ⟹ a" e "b ⟺ c" nella caratterizzazione dei campi vett. conservativi



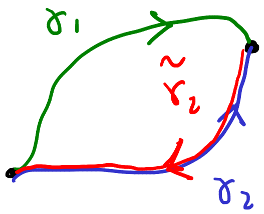
$f : A \to \mathbb{R} \qquad (\text{t.c. } \nabla f = F)$

$x \in A$, "insieme delle curve "buone" di estremi x_0 e x" $\neq \emptyset$

$f(x) := \displaystyle\int_\gamma F(P)\cdot dP$ dove γ è un qualsiasi elemento di (insieme sopra)

- f è ben posta per l'ipotesi (b)
- si dimostra che $\dfrac{\partial f}{\partial x_i} = F_i$

$$(b) \overset{??}{\Longleftrightarrow} (c)$$

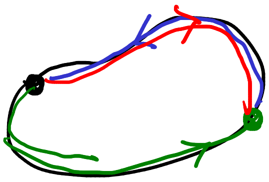


$$\int_{\gamma_1} F(P)\cdot dP - \int_{\gamma_2} F(P)\cdot dP =$$

$$\int_{\gamma_1} F(P)\cdot dP + \int_{\tilde{\gamma}_2} F(P)\cdot dP =$$

$$\oint_{\gamma_1 \cup \tilde{\gamma}_2} F(P)\cdot dP \overset{(c)}{=} 0$$

$$\Rightarrow \int_{\gamma_1} \cdots = \int_{\gamma_2} \cdots$$



$\cdots$