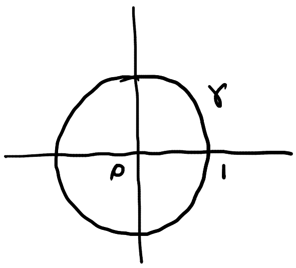
23 maggio 2025

Es. (l'integrale dipende dalla parametrizzazione)



$r(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$r'(t) = (-\sin t, \cos t)$

$\|r'(t)\| = 1$

$s(t) = (\cos t, \sin t) \quad t \in [0, 4\pi]$

$s'(t) = \dots$

$\|s'(t)\| = 1$

$f(x, y) = 1$

$$\int_{(\gamma, r)} f\, ds = \int_0^{2\pi} 1 \cdot 1\, dt = 2\pi$$

$$\neq$$

$$\int_{(\gamma, s)} f\, ds = \int_0^{4\pi} 1 \cdot 1\, dt = 4\pi$$

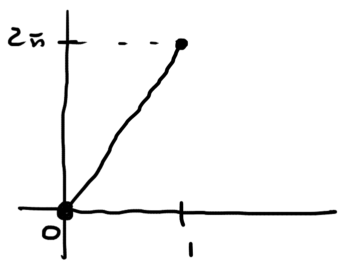
Es: (parametrizzazioni equivalenti e non equivalenti)

$r(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$s_1(t) = (\cos(2\pi t), \sin(2\pi t)) \quad t \in [0, 1]$

$= (\cos(h(t)), \sin(h(t))) = r(h(t))$

con $h(t) = 2\pi t$



strett. cresc.

$\Rightarrow$ s_1 equiv. a r

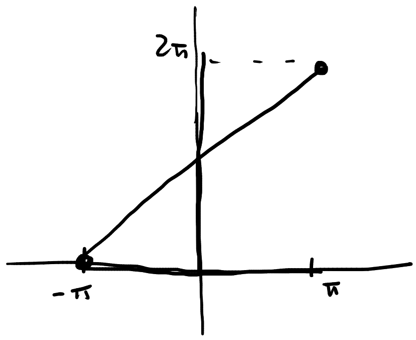
e induce su γ

lo stesso verso di percorrenza

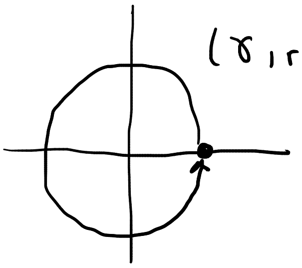
• $s_2(t) = (\cos(\pi + t), \sin(\pi + t)) \quad t \in [-\pi, \pi]$

$= r(\pi + t)$

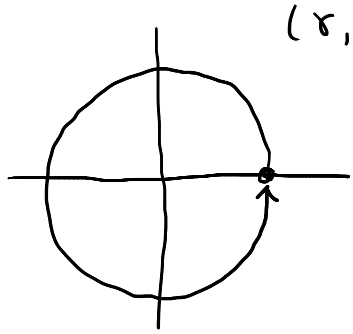
$h(t) = \pi + t$



(γ, r)

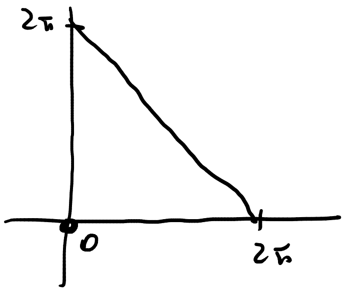


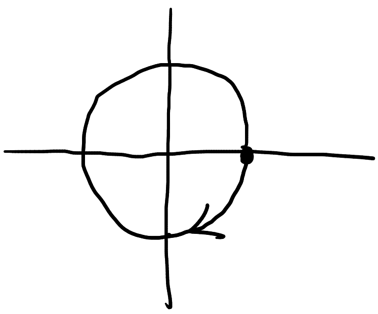
(γ, s_2)



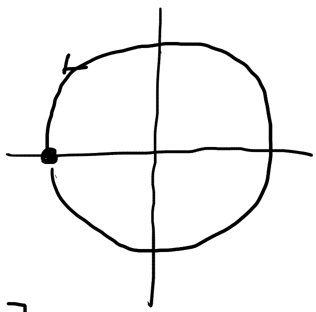
$s_3(t) = (\cos(2\pi - t), \sin(2\pi - t))$

$= r(h(t))$ $\quad h(t) = 2\pi - t \quad$ decresc.





$s_4(t) = (\cos t, \sin t) \quad t \in [-\pi, \pi]$



s_4 non è equiv. a r

$h(t) = t$? No! $\quad h([-\pi,\pi]) \neq [0, 2\pi]$

Se per assurdo esistesse $h: [-\pi, \pi] \to [0, 2\pi]$ bigettiva, di classe C^1, con $h'(t) \neq 0$ t.c.

$s_4(t) = r(h(t))$ avrei:

def. di s_4

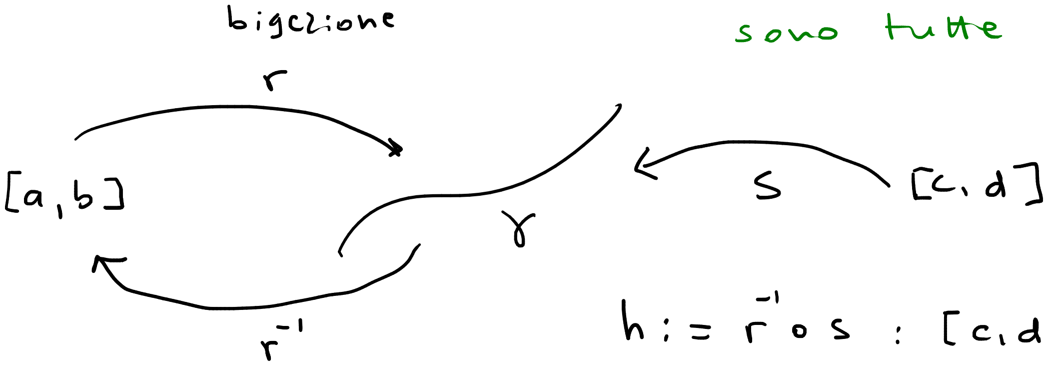
$s_4(-\pi) = (\cos(-\pi), \sin(-\pi)) = (-1, 0) \neq$

$\parallel$

$s_4(-\pi) = r(h(-\pi)) =$ $h\uparrow$: $r(0) = (\cos 0, \sin 0) = (1,0)$

$h\downarrow$: $r(2\pi) = (\cos 2\pi, \sin 2\pi) = (1,0)$

Motivazione del fatto che le parametrizzazioni "buone" sono tutte equivalenti



$$h := r^{-1} \circ s : [c,d] \to [a,b]$$

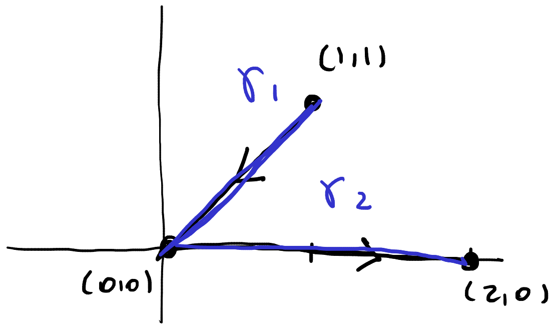
$$\Downarrow$$

$$\underbrace{r \circ h} = r \circ (r^{-1} \circ s) = (r \circ r^{-1}) \circ s = \underbrace{s}$$

Esempi (calcolo di integrali)

- $f(x,y) = e^{x+y}$ $\qquad$ $\mathrm{dom}(f) = \mathbb{R}^2 \Rightarrow \gamma \subset \mathrm{dom}(f)$
  f continua

poligonale di vertici $(1,1)$, $(0,0)$, $(2,0)$



poligonale: reg. a tratti

concatenamento del segmento cong. $(1,1)$ e $(0,0)$ e il segm. cong. $(0,0)$ e $(2,0)$

$$\int_\gamma f \, ds = \int_{\gamma_1} f \, ds + \int_{\gamma_2} f \, ds$$

γ_1 : $\qquad r(t) = (\overset{x}{\overbrace{t}}, \overset{y}{\overbrace{t}})$ $\qquad t \in [0,1]$

$\qquad r'(t) = (1, 1)$

$\qquad \|r'(t)\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\Rightarrow \int_{\gamma_1} f \, ds = \int_0^1 e^{t+t} \sqrt{2} \, dt = \sqrt{2} \left[\frac{e^{2t}}{2} \right]_0^1 = \frac{\sqrt{2}}{2} (e^2 - 1)$$

γ_2 : $\qquad r(t) = (\overset{x}{\overbrace{t}}, \overset{y}{\overbrace{0}})$ $\qquad t \in [0, 2]$

$r'(t) = (1,0)$

$\|r'(t)\| = 1$

$$\Rightarrow \int_{\gamma_2} f\,ds = \int_0^2 e^{t+0} \cdot 1\,dt = e^2 - 1$$

Quindi: $\displaystyle\int_\gamma f\,ds = \left(\frac{\sqrt{2}}{2} + 1\right)(e^2 - 1)$

$x \neq y \qquad r(t) = x + t(y - x) \qquad t \in [0,1]$

$$\gamma_1 : \quad r(t) = \underbrace{(1,1)}_{x} + t\,(\underbrace{(0,0)}_{y} - \underbrace{(1,1)}_{x})$$

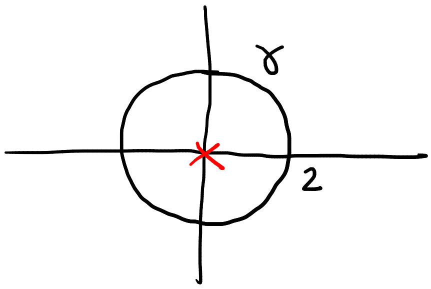
$$= (1,1) + t(-1,-1)$$

$$= (1-t,\ 1-t) \qquad t \in [0,1]$$

$$\gamma_2 : \quad r(t) = (0,0) + t\,((2,0) - (0,0)) \qquad t \in [0,1]$$

$$= (2t,\ 0)$$

- $f(x,y) = \dfrac{y^2}{x^2 + y^2}$



$\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

$\gamma \subset \mathrm{dom}(f)$, f continua

$$\gamma : \quad r(t) = (\overbrace{2\cos t}^{x},\ \overbrace{2\sin t}^{y}) \qquad t \in [0, 2\pi]$$

$r'(t) = (-2\sin t,\ 2\cos t)$

$\|r'(t)\| = 2$

$$\int_\gamma f\,ds = \int_0^{2\pi} \frac{4\sin^2 t}{4\cos^2 t + 4\sin^2 t} \cdot 2\,dt = \int_0^{2\pi} 2\sin^2 t\,dt$$

$$= \int_0^{2\pi} 2 \cdot \frac{1 - \cos 2t}{2}\,dt = 2\pi$$

Esempi (calcolo di lunghezze)

- $r(t) = (t + \sin t, \underbrace{\cos t}_{}^{\text{ingett.}})$ $\quad t \in [0, \pi]$

$r'(t) = (1 + \cos t, -\sin t)$

$\|r'(t)\| = \sqrt{1 + \cos^2 t + 2\cos t + \sin^2 t}$

$= \sqrt{2 + 2\cos t}$

$L(\gamma) = \int_0^{\pi} \sqrt{2\underbrace{(1+\cos t)}_{2\frac{1+\cos t}{2}\cdot 2}}\, dt = \int_0^{\pi} \sqrt{4\cos^2\left(\frac{t}{2}\right)}\, dt$

↑ curva semplice

$t \in [0, \pi]$

$\Rightarrow \frac{t}{2} \in [0, \frac{\pi}{2}]$

$\Rightarrow \cos\left(\frac{t}{2}\right) \geq 0$

$= \int_0^{\pi} 2\cos\left(\frac{t}{2}\right) dt = \dots$

- $r(t) = (2\cos t, 2\sin t, \underbrace{3t}_{\text{ing.}})$ $\quad t \in [0, 2\pi]$

$r'(t) = (-2\sin t, 2\cos t, 3)$

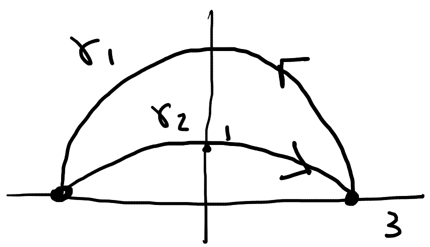
$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 9} = \sqrt{13}$

$L(\gamma) = \int_0^{2\pi} \sqrt{13}\, dt = 2\pi\sqrt{13}$

- Lunghezza del concatenamento di:

$\gamma_1 \quad : r(t) = (3\cos t, 3\sin t) \quad t \in [0, \pi]$

$\gamma_2 \quad :$ curva grafico di $f(x) = 1 - \frac{x^2}{9} \quad x \in [-3, 3]$



$r(t) = \left(t, 1 - \frac{t^2}{9}\right) \quad t \in [-3, 3]$

$r'(t) = \left(1, -\frac{2}{9}t\right)$

$$L(\gamma) = L(\gamma_1) + L(\gamma_2)$$

$$L(\gamma_1) = 3\pi$$

$$L(\gamma_2) = \int_{-3}^{3} \| \gamma'(t) \| \, dt = \int_{-3}^{3} \sqrt{1 + \frac{4}{81} t^2} \, dt$$

$$= 2 \int_{0}^{3} \sqrt{1 + \frac{4}{81} t^2} \, dt$$

$$\sqrt{1 + \frac{4}{81} t^2} = \frac{2}{9} t + \tau$$

$$1 + \cancel{\frac{4}{81} t^2} = \cancel{\frac{4}{81} t^2} + \frac{4}{9} t \tau + \tau^2$$

$$\frac{4}{9} t \tau = \tau^2 - 1$$

$$t = \frac{9}{4} \cdot \frac{\tau^2 - 1}{\tau}$$

$$dt = \ldots\ldots \, d\tau$$

...

(integrale di funzione razionale)