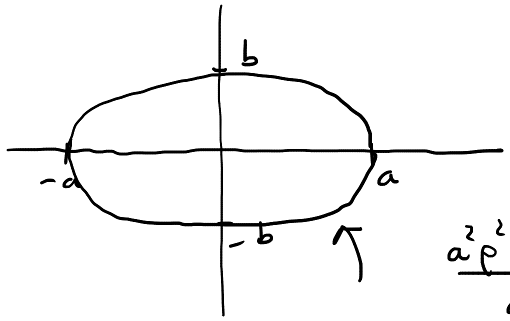
22 maggio 2025

Circonferenza: $x^2+y^2=r^2 \iff \frac{x^2}{r^2}+\frac{y^2}{r^2}=1$

Ellisse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a,b>0)$



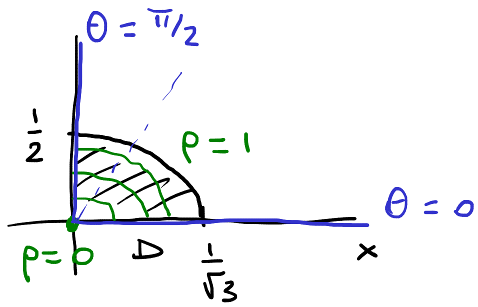
$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases}$$

$$\frac{a^2\rho^2\cos^2\theta}{a^2}+\frac{b^2\rho^2\sin^2\theta}{b^2}=1$$

$\rho^2=1 \qquad \rho=1$

Es. (coord. ellittiche)

- $f(x,y)=x+y$ $\qquad D:$



$3x^2+4y^2=1$

$$\frac{x^2}{\left(\frac{1}{\sqrt3}\right)^2}+\frac{y^2}{\left(\frac12\right)^2}=1 \qquad a=\frac{1}{\sqrt3},\ b=\frac12$$

Coord. ellittiche: $\Phi(\rho,\theta)=\Big(\overbrace{\frac{1}{\sqrt3}\rho\cos\theta}^{x},\ \overbrace{\frac12\rho\sin\theta}^{y}\Big)$

$$\det J_\Phi(\rho,\theta)=\frac{1}{2\sqrt3}\rho$$

$$\Phi^{-1}(D)=\left\{(\rho,\theta)\ \middle|\ 0\le\rho\le1,\ 0\le\theta\le\frac{\pi}{2}\right\}$$

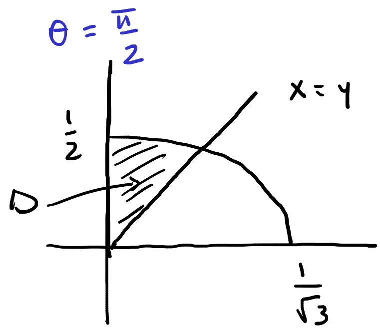
$$\iint_D f(x,y)\,dx\,dy=\iint_{\Phi^{-1}(D)} f(\Phi(\rho,\theta))\,|\det J_\Phi(\rho,\theta)|\,d\rho\,d\theta$$

$$=\iint_{[0,1]\times[0,\frac{\pi}{2}]}\left(\frac{1}{\sqrt3}\rho\cos\theta+\frac12\rho\sin\theta\right)\frac{1}{2\sqrt3}\rho\ d\rho\,d\theta$$

$$= \iint_{[0,1]\times[0,\frac{\pi}{2}]} \frac{1}{2\sqrt{3}}\rho^2 \left(\frac{1}{\sqrt{3}}\cos\theta + \frac{1}{2}\sin\theta \right) d\rho\, d\theta$$

$$= \int_0^1 \frac{1}{2\sqrt{3}}\rho^2\, d\rho \cdot \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{3}}\cos\theta + \frac{1}{2}\sin\theta \right) d\theta = \dots$$

- $f(x,y) = x+y$



Tutto come nell' esempio precedente, tranne

$$\Phi^{-1}(D) = \{(\rho,\theta) \mid 0 \le \rho \le 1,\ ? \le \theta \le \frac{\pi}{2}\}$$ NON $\frac{\pi}{4}$!!

$$\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases}$$

$x = y \iff a\rho\cos\theta = b\rho\sin\theta$

$\overset{\rho\neq 0}{\iff} a\cos\theta = b\sin\theta$

$\underset{(\cos\theta\neq 0)}{\iff} \tan\theta = \frac{a}{b}$

(In coord. polari: $a=b \Rightarrow \tan\theta = 1$

$\underbrace{\theta}_{\in(0,\frac{\pi}{2})} = \arctan(1) = \frac{\pi}{4}$)

Nell' esempio:

$$\begin{cases} x = \frac{1}{\sqrt{3}}\rho\cos\theta \\ y = \frac{1}{2}\rho\sin\theta \end{cases}$$

$x = y \iff \frac{1}{\sqrt{3}}\rho\cos\theta = \frac{1}{2}\rho\sin\theta$

$\iff \tan\theta = \frac{2}{\sqrt{3}}, \quad \theta\in(0,\frac{\pi}{2})$

$\iff \theta = \arctan\left(\frac{2}{\sqrt{3}}\right) =: \theta_0$

$$\Rightarrow \Phi^{-1}(D) = \{(\rho,\theta) \mid 0 \le \rho \le 1,\ \theta_0 \le \theta \le \frac{\pi}{2}\}$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{[0,1]\times[\theta_0,\frac{\pi}{2}]} f(\Phi(\rho,\theta))\, |\det J_\Phi(\rho,\theta)|\, d\rho\, d\theta$$

$$= \dots = \int_0^1 \frac{1}{2\sqrt{3}}\rho^2\, d\rho \cdot \underbrace{\int_{\theta_0}^{\pi/2} \left(\frac{1}{\sqrt{3}}\cos\theta + \frac{1}{2}\sin\theta \right) d\theta}$$

$$\| $$

$$\left[\frac{1}{\sqrt{3}}\sin\theta - \frac{1}{2}\cos\theta \right]_{\theta_0}^{\pi/2}$$

$$\|$$

$$\frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\sin\theta_0 - \frac{1}{2}\cos\theta_0 \right)$$

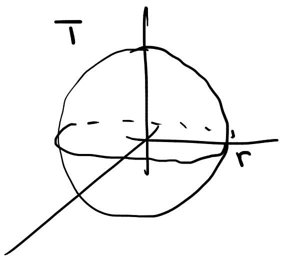
$$\left[\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ \tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta} \end{cases} \quad \dots \quad \begin{array}{l} \cos\theta = \pm \dfrac{1}{\sqrt{1+\tan^2\theta}} \\ \sin\theta = \pm \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} \end{array} \right]$$

$$\sin\theta_0 = + \frac{\tan\theta_0}{\sqrt{1+\tan^2\theta_0}} = \frac{\frac{2}{\sqrt{3}}}{\sqrt{1+\frac{4}{3}}} = \dots$$

$$\cos\theta_0 = \dots$$

Es. (coord. sferiche)

- volume della palla di centro $(0,0,0)$ e raggio r



$$\Phi(\rho,\varphi,\theta) = (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)$$

$$\det J_\Phi(\rho,\varphi,\theta) = \rho^2\sin\varphi$$

$$\Phi^{-1}(T) = \{(\rho,\varphi,\theta) \mid 0\le\rho\le r,\ 0\le\varphi\le\pi,\ 0\le\theta\le 2\pi\}$$

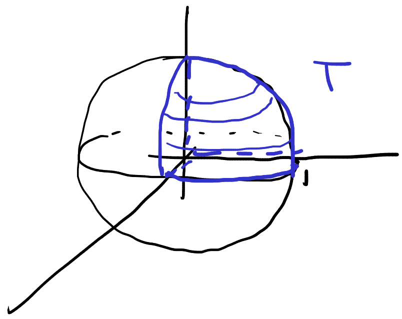
$$m_3(T) = \iiint_T 1\, dx\, dy\, dz = \iiint_{\Phi^{-1}(T)} 1\ \rho^2\sin\varphi\ d\rho\, d\varphi\, d\theta$$

$$= \iiint_{[0,r]\times[0,\pi]\times[0,2\pi]} \rho^2\sin\varphi\ d\rho\, d\varphi\, d\theta$$

$$= \int_0^r \rho^2 \, d\rho \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{2\pi} d\theta$$

$$= \left[\frac{\rho^3}{3} \right]_0^r \left[-\cos\varphi \right]_0^{\pi} 2\pi = \frac{2\pi}{3} r^3 \cdot 2 = \frac{4}{3}\pi r^3 \quad \checkmark$$

- $f(x,y,z) = \sqrt{x^2+y^2+z^2}$



$\Phi(\rho, \varphi, \theta) = \dots$

$\det J_\Phi(\rho,\varphi,\theta) = \rho^2 \sin\varphi$

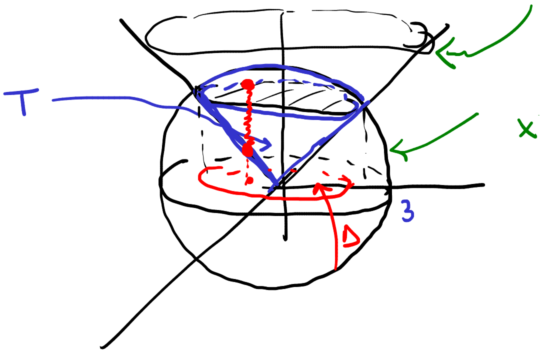
$$\Phi^{-1}(T) = \{ (\rho,\varphi,\theta) \mid 0 \le \rho \le 1, \; 0 \le \varphi \le \tfrac{\pi}{2}, \; 0 \le \theta \le \tfrac{\pi}{2} \}$$

$$\iiint_T f(x,y,z) \, dx\,dy\,dz = \iiint_{\Phi^{-1}(T)} f(\Phi(\rho,\varphi,\theta)) \, |\det J_\Phi(\rho,\varphi,\theta)| \, d\rho\, d\varphi\, d\theta$$

$$= \iiint_{[0,1] \times [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]} \rho \; \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$= \int_0^1 \rho^3 \, d\rho \int_0^{\pi/2} \sin\varphi \, d\varphi \int_0^{\pi/2} d\theta = \dots$$

- volume di



$$z = \sqrt{x^2+y^2}$$

$$x^2+y^2+z^2 = 9$$

$$z^2 = 9 - (x^2+y^2)$$

$$z = \sqrt{9-(x^2+y^2)}$$

$$m_3(T) = \iiint_T 1 \, dx\,dy\,dz = (*)$$

Descrivo T come insieme normale rispetto al piano xy :

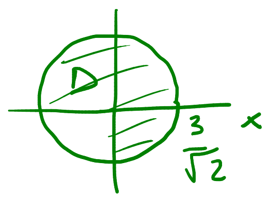
$$T = \{ (x,y,z) \mid (x,y) \in D, \ \sqrt{x^2+y^2} \le z \le \sqrt{9-(x^2+y^2)} \}$$

$\uparrow$ $\cancel{??}$

$$\begin{cases} x^2+y^2+z^2 = 9 \\ z = \sqrt{x^2+y^2} \end{cases} \qquad x^2+y^2+x^2+y^2 = 9 \quad \Rightarrow \quad x^2+y^2 = \frac{9}{2}$$

$$z^2 = x^2+y^2$$

$\Rightarrow$



per fili:

$$(*) = \iint_D \left(\int_{\sqrt{x^2+y^2}}^{\sqrt{9-(x^2+y^2)}} 1 \, dz \right) dx\,dy$$

$$= \iint_D \left(\sqrt{9-(x^2+y^2)} - \sqrt{x^2+y^2} \right) dx\,dy$$

coord. polari

$$= \iint_{[0,\frac{3}{\sqrt{2}}]\times[0,2\pi]} \left(\sqrt{9-\rho^2} - \rho \right) \rho \, d\rho\, d\theta$$

$$= 2\pi \left(\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-\rho^2}\, \rho \, d\rho - \underbrace{\int_0^{\frac{3}{\sqrt{2}}} \rho^2 \, d\rho}_{\checkmark} \right)$$

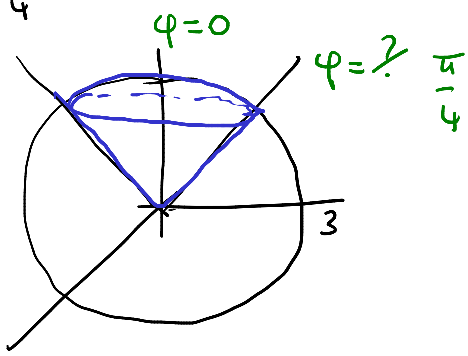
$$t = 9-\rho^2$$

$$dt = -2\rho \, d\rho$$

...

In alternativa, uso coord. sferiche

$$\Phi^{-1}(T) = \{ (\rho,\varphi,\theta) \mid 0 \le \rho \le 3, \ 0 \le \varphi \le \frac{\pi}{4}, \ 0 \le \theta \le 2\pi \}$$



$$z = \sqrt{x^2+y^2} \iff$$

$$\underline{\rho\cos\varphi} = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}$$

$$= \sqrt{\rho^2\sin^2\varphi} = \underline{\rho\sin\varphi}$$

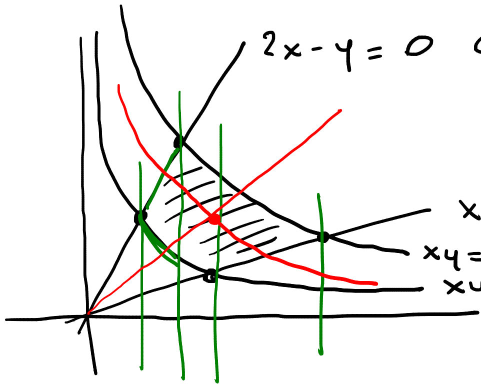
$\iff \cos\varphi = \sin\varphi \iff \varphi = \frac{\pi}{4}$

$$m_3(T) = \iiint_T 1\,dx\,dy\,dz = \iiint_{[0,3]\times[0,\frac{\pi}{4}]\times[0,2\pi]} 1\,\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^3 \rho^2\,d\rho \int_0^{\frac{\pi}{4}} \sin\varphi\,d\varphi \int_0^{2\pi} d\theta = \dots$$

Esempio (cambiamento di variabili "ad hoc")

$f(x,y) = x^2 y^2$



$$\left.\begin{array}{l} u = xy \\ v = \frac{y}{x} \end{array}\right\} =: \Phi^{-1}(x,y)$$

$x, y > 0 \quad (\Rightarrow u, v > 0)$

Determino Φ: $\begin{cases} xy = u \\ \frac{y}{x} = v \end{cases}$ $\begin{cases} v x^2 = u \\ y = v x \end{cases}$

$$x^2 = \frac{u}{v} \overset{x>0}{\Rightarrow} x = \sqrt{\frac{u}{v}} = u^{1/2} v^{-1/2}$$

$$y = v\,u^{1/2} v^{-1/2} = u^{1/2} v^{1/2}$$

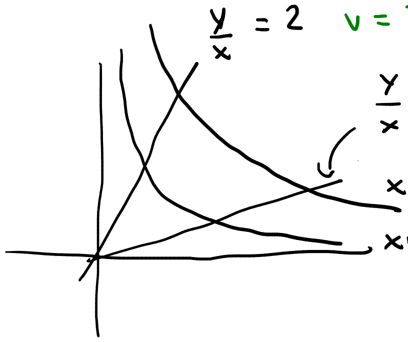
Quindi: $\Phi(u,v) = \left(u^{\frac{1}{2}} v^{-\frac{1}{2}},\ u^{\frac{1}{2}} v^{\frac{1}{2}}\right) \quad u, v > 0$

Φ di classe C^1 in $(0,+\infty)\times(0,+\infty)$

$$\det J_\Phi(u,v) = \det \begin{pmatrix} \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} & -\frac{1}{2} u^{\frac{1}{2}} v^{-\frac{3}{2}} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{pmatrix}$$

$$= \frac{1}{4} v^{-1} + \frac{1}{4} v^{-1} = \frac{1}{2v} \quad (\neq 0)$$

$$\Phi^{-1}(D) = \left\{ (u,v) \;\middle|\; 2 \le u \le 4, \quad \frac{1}{2} \le v \le 2 \right\}$$



$$\iint_D x^2 y^2 \, dx\, dy = \iint_{\Phi^{-1}(D)} \left(u^{\frac{1}{2}} v^{-\frac{1}{2}}\right)^2 \left(u^{\frac{1}{2}} v^{\frac{1}{2}}\right)^2 \frac{1}{2v} \, du\, dv$$

$$= \iint_{[2,4]\times[\frac{1}{2},2]} u v^{-1} u v \frac{1}{2v} \, du\, dv$$

$$= \int_2^4 u^2 \, du \int_{\frac{1}{2}}^{2} \frac{1}{2v} \, dv = \dots$$

Esempio (integrale curvilineo di campo scalare)

$$f(x,y,z) = xy + z \qquad A = \mathbb{R}^3$$

$$r(t) = (\overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t}) \qquad t \in [0, 4\pi]$$

$$r'(t) = (-\sin t, \cos t, 1)$$

$$\| r'(t) \| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\int_\gamma f \, ds = \int_0^{4\pi} ((\cos t)(\sin t) + t) \sqrt{2} \, dt = \dots$$