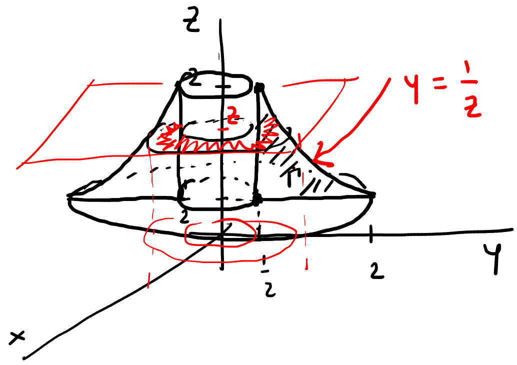
20 maggio 2025

Esempi (volume di solidi di rotazione)

- volume del solido T ottenuto ruotando intorno all' asse z l'insieme

$$\Gamma := \left\{ (0,y,z) \;\middle|\; \tfrac{1}{2} \le z \le 2 ,\ \tfrac{1}{2} \le y \le \tfrac{1}{z} \right\}$$



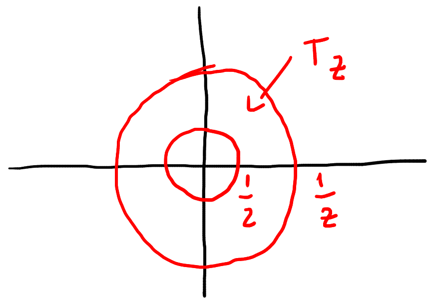
$$m_3(T) = \iiint_T 1 \, dx\,dy\,dz$$

per strati

$$= \int_{\frac{1}{2}}^{2} \left(\iint_{T_z} 1 \, dx\,dy \right) dz$$

$$= \int_{\frac{1}{2}}^{2} m_2(T_z)\, dz$$

?



$$= \int_{1/2}^{2} \pi \left(\frac{1}{z^2} - \frac{1}{2^2} \right) dz$$

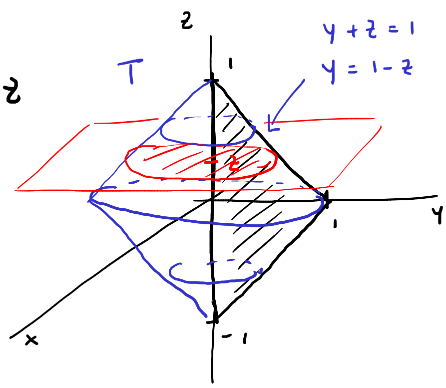
$$= \pi \left[-\frac{1}{z} - \frac{1}{4} z \right]_{\frac{1}{2}}^{2} = \dots$$

- volume del solido T ottenuto ruotando attorno all' asse z il triangolo di vertici $(0,0,1)$, $(0,0,-1)$, $(0,1,0)$

$$m_3(T) = \iiint_T 1 \, dx\,dy\,dz = \int_{-1}^{1} \left(\iint_{T_z} 1 \, dx\,dy \right) dz$$

SIMMETRIA

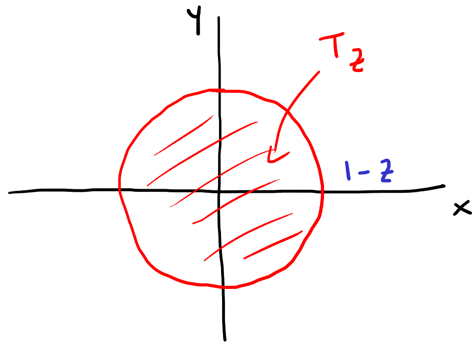
$$= 2 \int_0^1 \left(\iint_{T_z} 1 \, dx\,dy \right) dz$$



$$= 2 \int_0^1 m_2(T_z)\, dz$$

$$= 2\int_0^1 \pi (1-z)^2 \, dz$$

$$= -2\pi \left[\frac{(1-z)^3}{3} \right]_0^1 = \dots$$



Commento (sulla formula di cambiamento di variabili)

$f: [a,b] \to \mathbb{R}$ continua

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt =$$

$x = \varphi(t)$

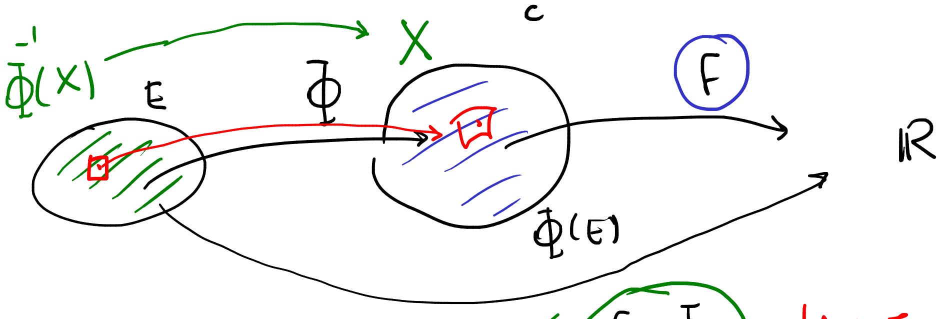
$(t = \varphi^{-1}(x))$

φ invertibile $\quad \varphi: [c,d] \to [a,b]$

φ cresc. $(\varphi' \geq 0)$ $\quad \displaystyle\int_c^d f(\varphi(t))\,|\varphi'(t)|\,dt$

φ decr. $(\varphi' \leq 0)$ $\quad \displaystyle\int_d^c f(\varphi(t))\,\varphi'(t)\,dt = \int_c^d f(\varphi(t))(-\varphi'(t))\,dt$

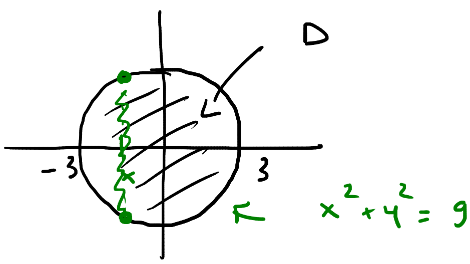
$$= \int_c^d f(\varphi(t))\,|\varphi'(t)|\,dt$$



$$(f \circ \Phi) \cdot |\det J_\Phi|$$

Esempi (coord. polari in $\mathbb{R}^2$)

- $f(x,y) = x^2 - 2y^2$



Integrando per verticali:

$$D = \{(x,y) \mid -3 \le x \le 3,\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}$$

$$\iint_D f(x,y)\,dx\,dy = \int_{-3}^{3} \left(\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2 - 2y^2)\,dy \right) dx$$

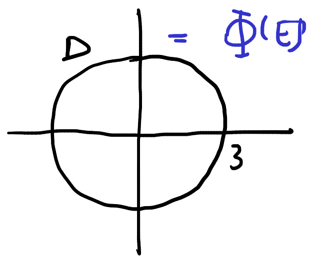
$$= \int_{-3}^{3} \left[x^2 y - \frac{2}{3} y^3 \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} \left(x^2 (2\sqrt{9-x^2}) - \frac{2}{3} \left((9-x^2)\sqrt{9-x^2} \cdot 2 \right) \right) dx$$

...

Risolvo con coord. polari di centro $(0,0)$

$$\Phi(\rho,\theta) = (\rho\cos\theta,\ \rho\sin\theta)$$



$= \Phi(E)$

$E = ?$

$(\Phi^{-1}(D) = ?)$

$$\Phi^{-1}(D) = \{(\rho,\theta) \mid 0 \le \rho \le 3,\ 0 \le \theta \le 2\pi\}$$

$$= [0,3] \times [0,2\pi]$$

$$\underset{\substack{\| \\ \Phi(E)}}{\iint_D} f(x,y)\,dx\,dy = \underset{\substack{\| \\ E}}{\iint_{\Phi^{-1}(D)}} f(\Phi(\rho,\theta))\,|\det J_\Phi(\rho,\theta)|\,d\rho\,d\theta$$

$$= \iint_{[0,3]\times[0,2\pi]} f(\rho\cos\theta,\ \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

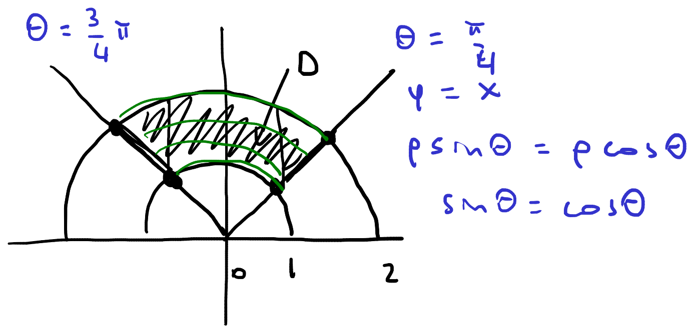
$$= \iint_{[0,3]\times[0,2\pi]} (\rho^2\cos^2\theta - 2\rho^2\sin^2\theta)\,\rho\,d\rho\,d\theta$$

$$= \iint_{[0,3]\times[0,2\pi]} \rho^3 (\cos^2\theta - 2\sin^2\theta)\, d\rho\, d\theta$$

$$= \underbrace{\int_0^3 \rho^3 d\rho}_{\text{immediato!}} \int_0^{2\pi} (\cos^2\theta - 2\sin^2\theta)\, d\theta = \dots$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ (facile)

- $f(x,y) = x^2 - 2y^2$



$$\Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$$

$$\Phi^{-1}(D) = \{(\rho,\theta) \mid 1 \le \rho \le 2,\ \frac{\pi}{4} \le \theta \le \frac{3}{4}\pi\}$$

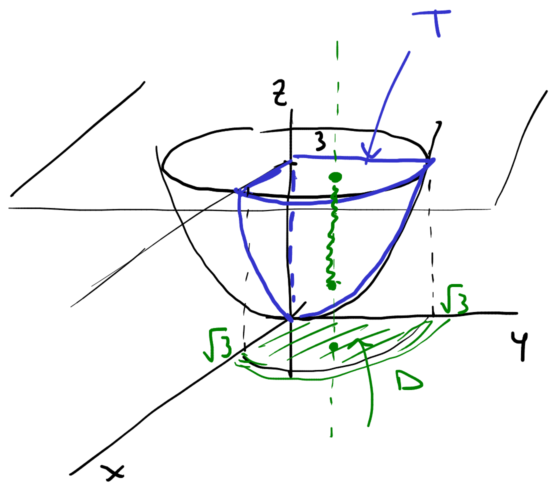
$$\iint_D f(x,y)\, dx\, dy = \iint_{\Phi^{-1}(D)} f(\Phi(\rho,\theta))\, |\det J_\Phi(\rho,\theta)|\, d\rho\, d\theta$$

$$= \iint_{[1,2]\times[\frac{\pi}{4},\frac{3}{4}\pi]} (\rho^2\cos^2\theta - 2\rho^2\sin^2\theta)\, \rho\, d\rho\, d\theta$$

$$= \int_1^2 \rho^3 d\rho \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (\cos^2\theta - 2\sin^2\theta)\, d\theta = \dots$$

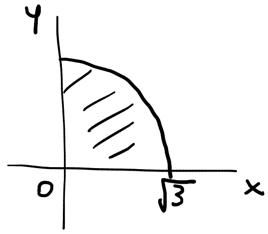
- $f(x,y,z) = xy$

T contenuto nel 1° ottante, delimitato da $x=0$, $y=0$, $z=3$ e $z = x^2+y^2$.



Integro per fili:

$$T = \{(x,y,z) \mid (x,y)\in D,\ x^2+y^2 \le z \le 3\}$$

con $D =$ 

$$\begin{cases} z = x^2+y^2 \\ z = 3 \end{cases}$$

$x^2+y^2 = 3$

$$\iiint_T f(x,y,z)\,dx\,dy\,dz =$$

$$\iint_D \left(\int_{x^2+y^2}^{3} xy\,dz \right) dx\,dy =$$

$$\iint_D xy\,(3-x^2-y^2)\,dx\,dy =$$

In coord. polari

$\Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$

$\Phi^{-1}(D) = [0,\sqrt{3}] \times [0,\frac{\pi}{2}]$

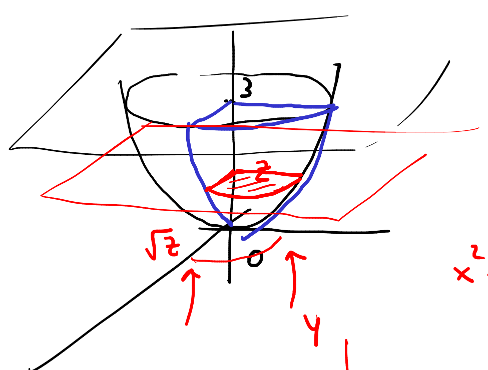
$$\iint_{[0,\sqrt{3}]\times[0,\frac{\pi}{2}]} \rho\cos\theta\,\rho\sin\theta\,(3-(\rho^2\cos^2\theta+\rho^2\sin^2\theta))\,\rho\,d\rho\,d\theta =$$

$$\iint_{[0,\sqrt{3}]\times[0,\frac{\pi}{2}]} \rho^3\cos\theta\sin\theta\,(3-\rho^2)\,d\rho\,d\theta =$$

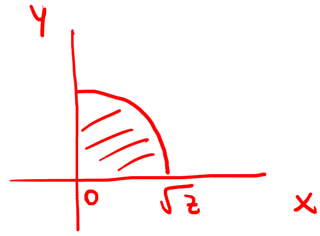
$$\int_0^{\sqrt{3}} \rho^3(3-\rho^2)\,d\rho \int_0^{\pi/2} \cos\theta\sin\theta\,d\theta =$$

$$\left[\frac{3}{4}\rho^4 - \frac{\rho^6}{6}\right]_0^{\sqrt{3}} \cdot \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \ldots$$

Integro per strati



$x^2+y^2 = z$



$$\iiint_T f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^3 \underbrace{\left(\iint_{T_z} xy\,dx\,dy \right)}_{=:\,I_z} dz$$

Con coord. polari:

$$I_z = \iint_{[0,\sqrt{z}]\times[0,\frac{\pi}{2}]} \rho\cos\theta \, \rho\sin\theta \, \rho \, d\rho \, d\theta$$

$$= \int_0^{\sqrt{z}} \rho^3 d\rho \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \left[\frac{\rho^4}{4}\right]_0^{\sqrt{z}} \left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2} = \frac{z^2}{4}\cdot\frac{1}{2} = \frac{z^2}{8}$$

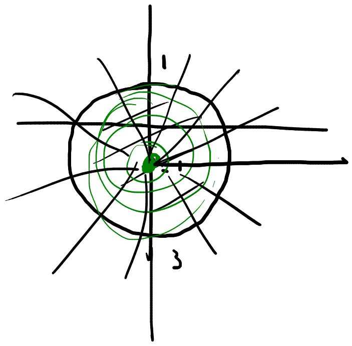
$$\Rightarrow \iiint_T f(x,y,z)\,dx\,dy\,dz = \int_0^3 \frac{z^2}{8}\,dz = \dots$$

- volume del solido T delimitato da $z = x^2+y^2$ e $z = 3-2y$

Già descritto come insieme normale:

$$T = \{(x,y,z) \mid (x,y)\in D,\ x^2+y^2 \le z \le 3-2y\}$$

con D: disco di centro $(0,-1)$ e raggio 2



$$m_3(T) = \iint_D \big(\underbrace{3-2y-(x^2+y^2)}_{4-x^2-(y+1)^2}\big)\,dx\,dy = (*)$$

Considero coord. polari di centro $(0,-1)$:

$$\Phi(\rho,\theta) = (\rho\cos\theta,\ -1+\rho\sin\theta)$$

$$\Rightarrow \Phi^{-1}(D) = \{(\rho,\theta) \mid 0\le\rho\le 2,\ 0\le\theta\le 2\pi\}$$

$$(*) = \iint_{[0,2]\times[0,2\pi]} \left(4-\rho^2\cos^2\theta - (-1+\rho\sin\theta+1)^2\right)\rho \; d\rho\, d\theta$$

$$= \iint_{[0,2]\times[0,2\pi]} (4-\rho^2\cos^2\theta-\rho^2\sin^2\theta)\rho \; d\rho\, d\theta$$

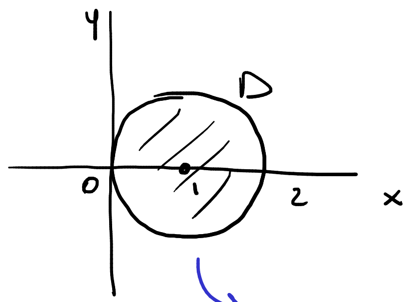
$$= \iint_{[0,2]\times[0,2\pi]} (4-\rho^2)\rho \; d\rho\, d\theta$$

$$= \int_0^2 (4-\rho^2)\rho\, d\rho \cdot \int_0^{2\pi} d\theta = 2\pi\left[2\rho^2-\frac{\rho^4}{4}\right]_0^2 = \dots$$

- $f(x,y)=\sqrt{x^2+y^2}$

$\downarrow$

coord. pol.
di centro $(0,0)$



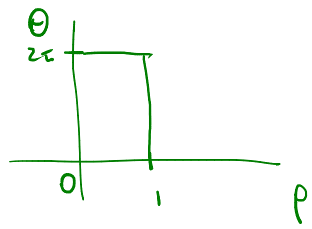
$\hookrightarrow$ coord. pol.
di centro $(1,0)$

Con coord. pol. di centro $(1,0)$:

$$\Phi(\rho,\theta) = (1+\rho\cos\theta,\; \rho\sin\theta)$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{[0,1]\times[0,2\pi]} \sqrt{(1+\rho\cos\theta)^2+\rho^2\sin^2\theta}\;\rho\; d\rho\, d\theta$$

$$= \iint_{[0,1]\times[0,2\pi]} \sqrt{1+2\rho\cos\theta+\rho^2}\;\rho\; d\rho\, d\theta$$



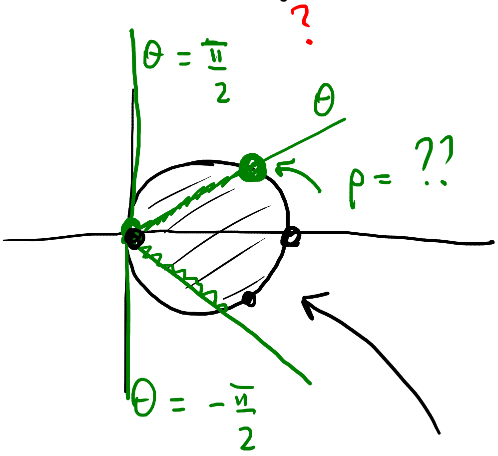
$$\cancel{=} \int_0^1 \left(\int_0^{2\pi} \underbrace{\sqrt{1+2\rho\cos\theta+\rho^2}\;\rho\; d\theta}_{???} \right) d\rho$$

$$= \int_0^{2\pi} \left(\int_0^1 \underbrace{\sqrt{1+2\rho\cos\theta+\rho^2}\;\rho\; d\rho}_{???} \right) d\theta$$

Provo con coord. polari di centro (0,0)

$$\iint_D f(x,y)\,dx\,dy = \iint_{?} \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}\;\rho\;d\rho\,d\theta$$

$$= \iint_{?} \rho^2\,d\rho\,d\theta$$



$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le \rho$$

$$(x-1)^2 + y^2 = 1$$
$$x^2 - 2x + 1 + y^2 = 1$$
$$x^2 + y^2 - 2x = 0$$

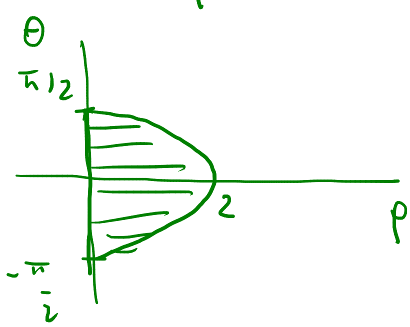
"Traduco": $\rho^2\cos^2\theta + \rho^2\sin^2\theta - 2\rho\cos\theta = 0$

$$\rho^2 - 2\rho\cos\theta = 0$$

$$\rho(\rho - 2\cos\theta) = 0 \quad \begin{cases} \rho = 0 \\ \rho = 2\cos\theta \end{cases}$$

$$\Phi^{-1}(D) = \left\{ (\rho,\theta) \;\middle|\; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\; 0 \le \rho \le 2\cos\theta \right\}$$

insieme normale nel piano (ρ,θ) rispetto all'asse θ



Quindi:

$$\iint_D f(x,y)\,dx\,dy = \iint_{\Phi^{-1}(D)} \rho^2\,d\rho\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} \rho^2\,d\rho \right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta\,d\theta$$

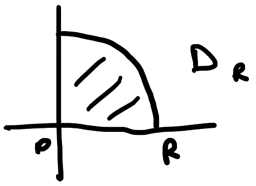
$$= \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\,d\theta \underset{\text{sostit.}}{=} \dots$$

Per soddisfare la curiosità di alcuni studenti presenti alla lezione, ricopio qui gli appunti dello scorso anno accademico sull'applicazione delle coordinate polari al calcolo dell'"integrale gaussiano"

$$\int_{-\infty}^{+\infty} \underbrace{e^{-t^2}}_{\text{pari}} dt = 2\int_0^{+\infty} e^{-t^2} dt = 2 \lim_{R \to +\infty} \int_0^R e^{-t^2} dt$$

non ho primitiva (esprimibile ...)

$\forall R > 0$:

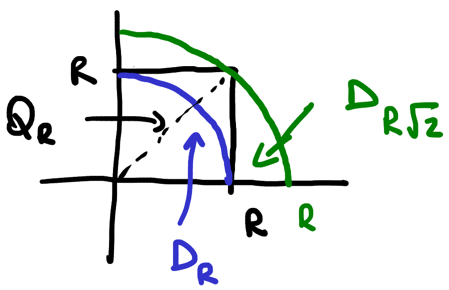


$$\iint_{D_R} e^{-(x^2+y^2)} dx\,dy \underset{\substack{\uparrow \\ \text{coord.} \\ \text{polari}}}{=} \iint_{[0,R]\times[0,\frac{\pi}{2}]} e^{-\rho^2} \rho \, d\rho$$

$$= \int_0^R e^{-\rho^2} \rho \, d\rho \cdot \int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{2} \left[-\frac{e^{-\rho^2}}{2} \right]_0^R$$

$$= \frac{\pi}{2} \cdot \frac{1 - e^{-R^2}}{2}$$

$$\Rightarrow \lim_{R \to +\infty} \iint_{D_R} e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}$$



$$D_R \subseteq Q_R \subseteq D_{R\sqrt{2}}$$

$\forall R > 0$:

$$\Rightarrow \underbrace{\iint_{D_R} e^{-(x^2+y^2)}\,dx\,dy}_{\downarrow\ \frac{\pi}{4}} \le \iint_{Q_R} e^{-(x^2+y^2)}\,dx\,dy \le \underbrace{\iint_{D_{R\sqrt{2}}} e^{-(x^2+y^2)}\,dx\,dy}_{\downarrow\ \frac{\pi}{4}}$$

$R \to +\infty$

$$\overset{TCO}{\Rightarrow} \quad \lim_{R\to+\infty} \iint_{Q_R} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$$

Oss:

$$\iint_{Q_R} e^{-(x^2+y^2)}\,dx\,dy = \iint_{[0,R]\times[0,R]} e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \int_0^R e^{-x^2}\,dx \cdot \int_0^R e^{-y^2}\,dy = \left(\int_0^R e^{-t^2}\,dt\right)^2$$

Dato che: $\left(\int_0^R e^{-t^2}\,dt\right)^2 \underset{R\to+\infty}{\longrightarrow} \frac{\pi}{4}$,

deduco $\quad \int_0^R e^{-t^2}\,dt \longrightarrow \frac{\sqrt{\pi}}{2}$

Quindi:

$$\boxed{\int_{-\infty}^{+\infty} e^{-t^2}\,dt} = 2 \lim_{R\to+\infty} \int_0^R e^{-t^2}\,dt = \boxed{\sqrt{\pi}}$$