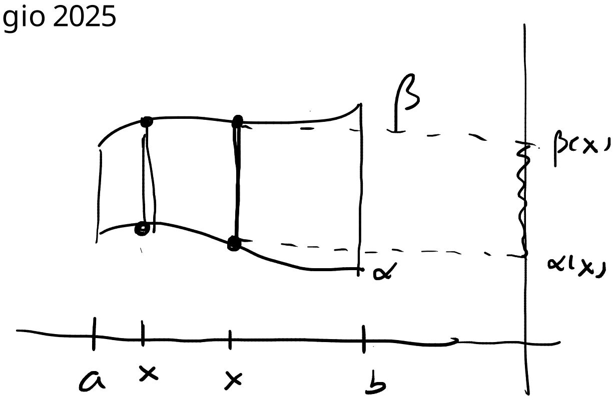
13 maggio 2025



$$\underbrace{\int_{\alpha(x)}^{\beta(x)} \overbrace{f(x,y)}\, dy}_{\text{dipende da } x}$$

$$\varphi := x \in [a,b] \longmapsto \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy \in \mathbb{R}$$

è continua (va dimostrato)

$\Rightarrow$ ha senso

$$\int_a^b \varphi(x)\, dx \quad \left(= \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy \right) dx \right)$$

$$\parallel$$

$$\iint_D f(x,y)\, dx\, dy$$

(va dimostrato)

Esempi (integrazione per verticali / orizzontali)

- $D = \{ (x,y) \mid a \le x \le b,\ \alpha(x) \le y \le \beta(x) \}$

continue in $[a,b]$

oss.

$$m_2(D) = \iint_D 1\, dx\, dy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} 1\, dy \right) dx$$

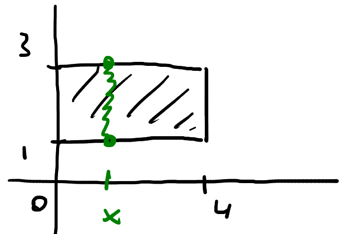
integr. per verticali

$$= \int_a^b (\beta(x) - \alpha(x))\, dx$$

coerente con la definizione data!

- $f(x,y) = x^2 + xy \qquad D = [0,4] \times [1,3]$

Scrivo D come normale rispetto all' asse x:



$$D = \{ (x,y) \mid \underset{a}{0} \le x \le \underset{b}{4}, \quad \underset{\alpha(x)}{1} \le y \le \underset{\beta(x)}{3} \}$$

$$\iint_D f(x,y)\, dx\, dy = \int_0^4 \left(\int_1^3 (x^2 + xy)\, dy \right) dx$$

$$= \int_0^4 \left[x^2 y + x \frac{y^2}{2} \right]_1^3 dx = \int_0^4 \left(3x^2 + \frac{9}{2} x - x^2 - \frac{x}{2} \right) dx$$

$$= \int_0^4 (2x^2 + 4x)\, dx = \left[\frac{2}{3} x^3 + 2x^2 \right]_0^4 = \frac{128}{3} + 32 = \frac{224}{3}$$

Se scrivo D come normale rispetto all' asse y:

$$D = \{ (x,y) \mid \underset{a}{1} \le y \le \underset{b}{3}, \quad \underset{\alpha(x)}{0} \le x \le \underset{\beta(x)}{4} \}$$

$$\iint_D f(x,y)\, dx\, dy = \int_1^3 \left(\int_0^4 (x^2 + xy)\, dx \right) dy$$

$$= \int_1^3 \left[\frac{x^3}{3} + \frac{x^2}{2} y \right]_0^4 dy = \int_1^3 \left(\frac{64}{3} + 8y \right) dy$$

$$= \left[\frac{64}{3} y + 4y^2 \right]_1^3 = \frac{64}{3} \cdot 2 + 4 \cdot 8 = \frac{128}{3} + 32 = \ldots \checkmark$$

Oss: $D = [a,b] \times [c,d]$

$$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx \underset{\uparrow}{=} \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

formula di inversione dell' ordine di integrazione

• $f(x,y) = x y^2 \qquad D = [0,4] \times [1,3]$

Integro per verticali:

$$\iint_D f(x,y)\,dx\,dy = \int_0^4 \Big(\int_1^3 \underbrace{x}_{\text{non dipende da } y} y^2\,dy \Big)\,dx$$

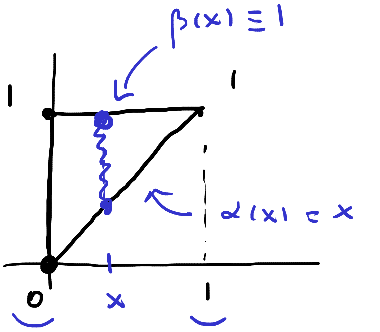
$$= \int_0^4 x \Big(\underbrace{\int_1^3 y^2\,dy}_{\text{non dipende da } x} \Big)\,dx = \int_0^4 x\,dx \int_1^3 y^2\,dy$$

$$= \Big[\frac{x^2}{2} \Big]_0^4 \cdot \Big[\frac{y^3}{3} \Big]_1^3 = 8 \cdot \frac{26}{3} = \frac{208}{3}$$

"a variabili separabili"

Oss: $$\iint_{\underbrace{[a,b]\times[c,d]}_{\text{rettangolo}}} g(x)\,h(y)\,dx\,dy = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

• $f(x,y) = x y^2$

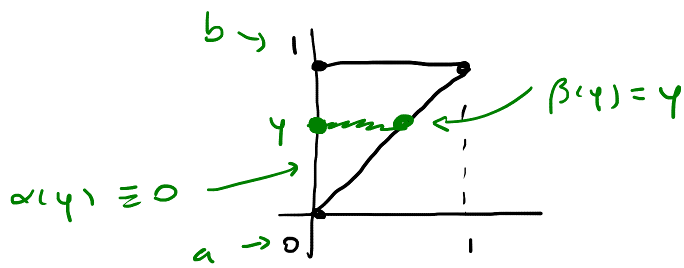


Per verticali:

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big(\int_x^1 x y^2\,dy \Big)\,dx = \int_0^1 \Big[x \frac{y^3}{3} \Big]_x^1 dx$$

$$= \int_0^1 \Big(\frac{x}{3} - \frac{x^4}{3} \Big) dx = \Big[\frac{x^2}{6} - \frac{x^5}{15} \Big]_0^1 = \frac{1}{6} - \frac{1}{15} = \frac{1}{10}$$

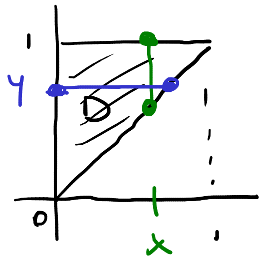
Integro per orizzontali:



$$\iint_D f(x,y)\,dx\,dy =$$

$$= \int_0^1 \Big(\int_0^y x y^2\,dx \Big) dy = \int_0^1 \Big[\frac{x^2}{2} y^2 \Big]_0^y dy = \int_0^1 \frac{y^4}{2}\,dy = \Big[\frac{y^5}{10} \Big]_0^1 = \frac{1}{10}$$

• $f(x,y) = e^{y^2}$



• Per verticali:

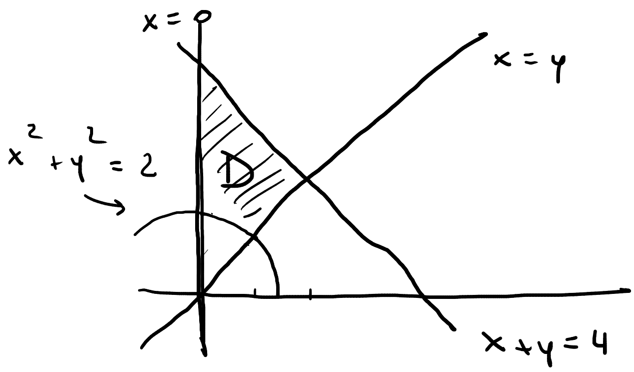
$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big(\underbrace{\int_x^1 e^{y^2}\,dy}_{} \Big)\,dx \qquad ?!!$$

• Per orizzontali:

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \Big(\int_0^y e^{y^2}\,dx \Big)\,dy = \int_0^1 e^{y^2}\,y\,dy$$

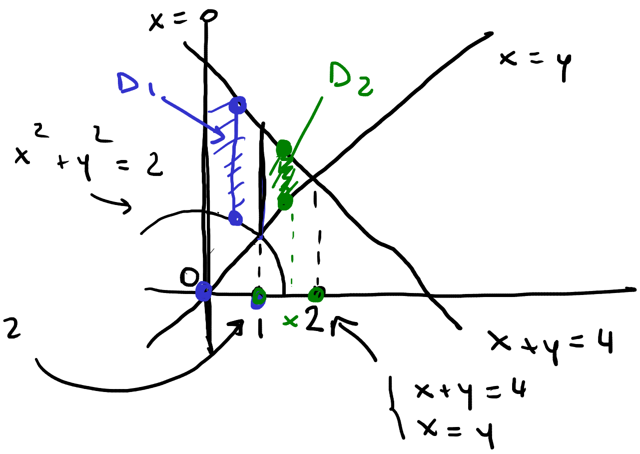
$$= \frac{1}{2}\int_0^1 e^{y^2}(2y)\,dy = \frac{1}{2}\left[e^{y^2}\right]_0^1 = \frac{e-1}{2}$$

• $f(x,y) = \dfrac{x}{y}$



$D_1 = \{(x,y) \mid 0 \le x \le 1,$
$\sqrt{2-x^2} \le y \le 4-x\}$

$D_2 = \{(x,y) \mid 1 \le x \le 2,$
$x \le y \le 4-x\}$ $\quad \begin{cases} x^2+y^2 = 2 \\ x = y \end{cases}$



Oss: $D = D_1 \cup D_2$

D_1 e D_2 non hanno punti interni in comune

$$\Rightarrow \underbrace{\iint_D f(x,y)\,dx\,dy}_{=:\,I} \underset{\text{additività}}{=} \underbrace{\iint_{D_1} f(x,y)\,dx\,dy}_{=:\,I_1} + \underbrace{\iint_{D_2} f(x,y)\,dx\,dy}_{=:\,I_2}$$

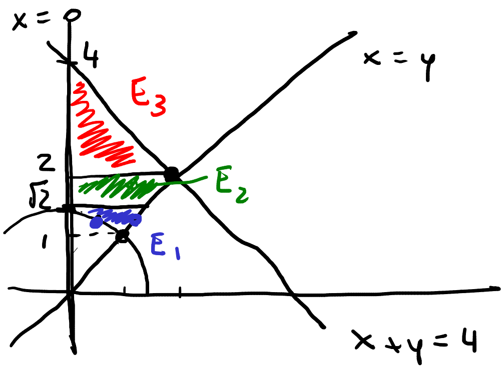
$$J_1 = \int_0^1 \left(\int_{\sqrt{2-x^2}}^{4-x} \frac{x}{y}\, dy \right) dx = \int_0^1 \left[x \ln y \right]_{\sqrt{2-x^2}}^{4-x} dx$$

$$= \int_0^1 \Big(\underbrace{x \ln(4-x)}_{\text{per parti}} - \frac{x}{2} \underbrace{\ln(2-x^2)}_{\text{per parti}} \Big)\, dx = \dots$$

(oppure: sostituzione + per parti)

$$I_2 = \int_1^2 \left(\int_x^{4-x} \frac{x}{y}\, dy \right) dx = \int_1^2 \left[x \ln y \right]_x^{4-x} dx$$

$$= \int_1^2 \Big(\underbrace{x \ln(4-x)}_{\text{come sopra}} - x \underbrace{\ln x}_{\text{per parti}} \Big)\, dx = \dots$$



$$E_1 = \{ (x,y) \mid 1 \le y \le \sqrt{2},\ \sqrt{2-y^2} \le x \le y \}$$

$$E_2 = \{ (x,y) \mid \sqrt{2} \le y \le 2,\ 0 \le x \le y \}$$

$$E_3 = \{ (x,y) \mid 2 \le y \le 4,\ 0 \le x \le 4-y \}$$

Oss: $E_1 \cup E_2 \cup E_3 = D$

E_1, E_2, E_3 privi di punti interni in comune:

$$\Rightarrow I = \underbrace{\iint_{E_1} \frac{x}{y}\, dx\, dy}_{J_1} + \underbrace{\iint_{E_2} \frac{x}{y}\, dx\, dy}_{J_2} + \underbrace{\iint_{E_3} \frac{x}{y}\, dx\, dy}_{J_3}$$

$$J_1 = \int_1^{\sqrt{2}} \left(\int_{\sqrt{2-y^2}}^{y} \frac{x}{y}\, dx \right) dy = \int_1^{\sqrt{2}} \left[\frac{1}{y} \frac{x^2}{2} \right]_{\sqrt{2-y^2}}^{y} dy$$

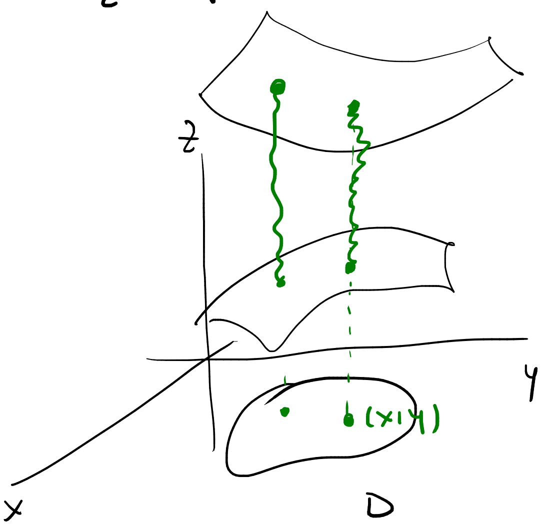
$$= \int_1^{\sqrt{2}} \left(\frac{y}{2} - \frac{2-y^2}{2y} \right) dy = \int_1^{\sqrt{2}} \left(\frac{y}{2} - \frac{1}{y} + \frac{y}{2} \right) dy = \int_1^{\sqrt{2}} \left(y + \frac{1}{y} \right) dy$$

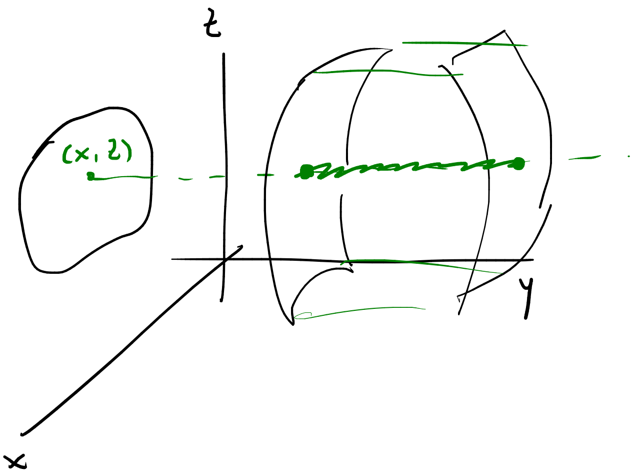
facile!

$$J_2 = \int_{\sqrt{2}}^{2} \left(\int_0^{y} \frac{x}{y}\, dx \right) dy = \int_{\sqrt{2}}^{2} \left[\frac{x^2}{2y} \right]_0^{y} dy = \int_{\sqrt{2}}^{2} \frac{y}{2}\, dy = \dots$$

$$J_3 = \int_2^4 \left(\int_0^{4-y} \frac{x}{y}\, dx \right) dy = \int_2^4 \left[\frac{x^2}{2y} \right]_0^{4-y} dy = \int_2^4 \frac{(4-y)^2}{2y}\, dy$$

$$= \int_2^4 \left(\frac{8}{y} - 4 + \frac{y}{2} \right) dy = \dots$$





Esempi (integrazione per fili)

* $T = \{ (x,y,z) \mid \underbrace{(x,y) \in D}_{\text{insieme normale in } \mathbb{R}^2},\ \underbrace{\gamma(x,y) \le z \le \delta(x,y)}_{\text{continue in } D} \}$

$$m_3(T) \overset{\text{oss.}}{=} \iiint_T 1\, dx\, dy\, dz$$

$$= \iint_D \left(\int_{\gamma(x,y)}^{\delta(x,y)} 1\, dz \right) dx\, dy$$

$$= \iint_D \left(\delta(x,y) - \gamma(x,y) \right) dx\, dy$$

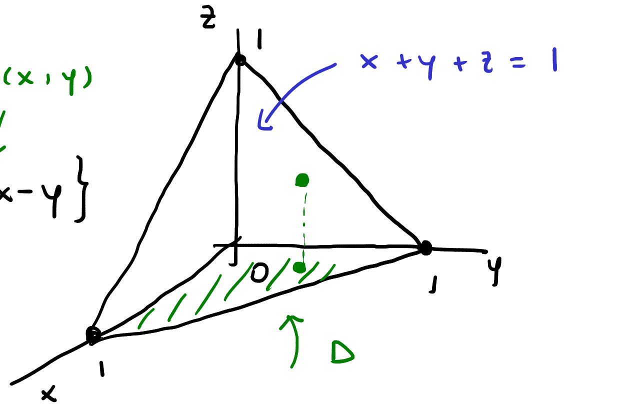
coerente con la definizione data

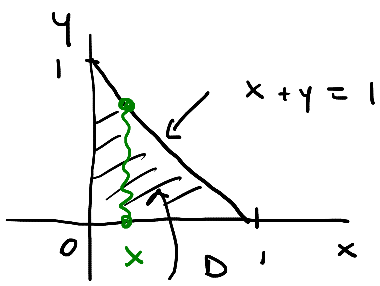
1. • $f(x,y,z) = x + z$

T : tetraedro di vertici $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$

Già fatto!

$$T = \{(x,y,z) \mid (x,y) \in D,\ \overset{\gamma(x,y)}{0} \le z \le \overset{\delta(x,y)}{1-x-y}\}$$





Integro per fili paralleli all'asse z:

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_0^{1-x-y} (x+z)\,dz \right) dx\,dy$$

$$= \iint_D \left[xz + \frac{z^2}{2} \right]_0^{1-x-y} dx\,dy = \iint_D \left(x(1-x-y) + \frac{(1-x-y)^2}{2} \right) dx\,dy$$

integro per vertic.

$$= \int_0^1 \left(\int_0^{1-x} \left(x(1-x-y) + \frac{(1-x-y)^2}{2} \right) dy \right) dx$$

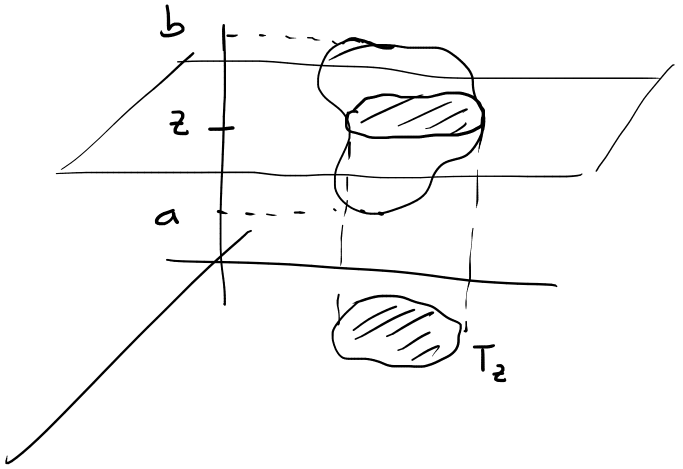
$$= \int_0^1 \left[-x\frac{(1-x-y)^2}{2} - \frac{(1-x-y)^3}{6} \right]_0^{1-x} dx$$

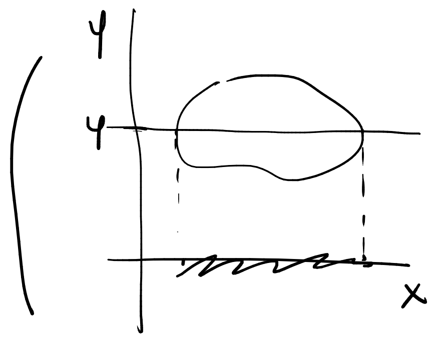
$$= \int_0^1 \left(x\frac{(1-x)^2}{2} + \frac{(1-x)^3}{6} \right) dx$$

$$= \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2} + \frac{(1-x)^3}{6} \right) dx$$

$$= \left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} - \frac{(1-x)^4}{24} \right]_0^1 = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} + \frac{1}{24}$$

$$= \frac{6-8+3+1}{24} = \frac{1}{12}$$

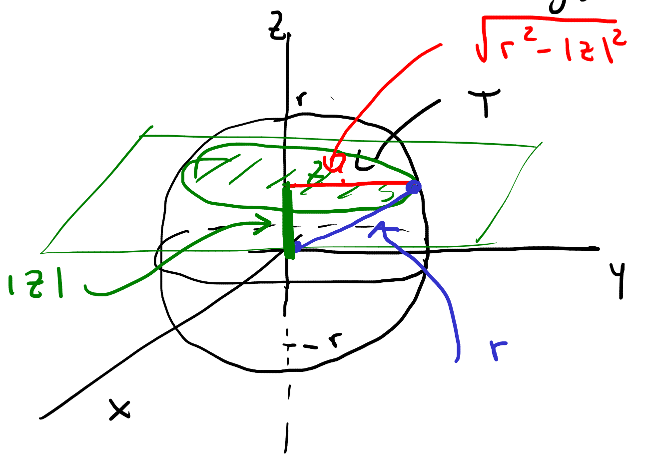




Esempi (integrazione per strati)

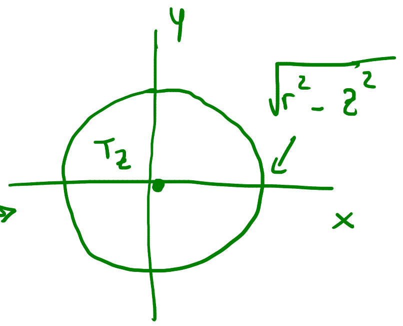
- volume della palla di centro 0 e raggio r

$$m_3(T) = \iiint_T 1 \, dx \, dy \, dz$$



$a = -r, \quad b = r$

$T_z =$



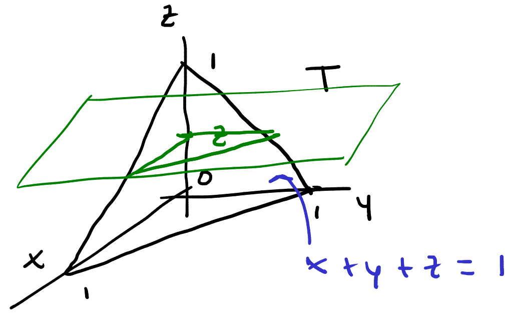
area di $T_z = \pi(r^2 - z^2)$

$$m_3(T) = \iiint_T 1 \, dx \, dy \, dz = \int_{-r}^{r} \left(\iint_{T_z} 1 \, dx \, dy \right) dz$$

$$= \int_{-r}^{r} \pi (r^2 - z^2) \, dz = 2\pi \int_0^r (r^2 - z^2) \, dz$$

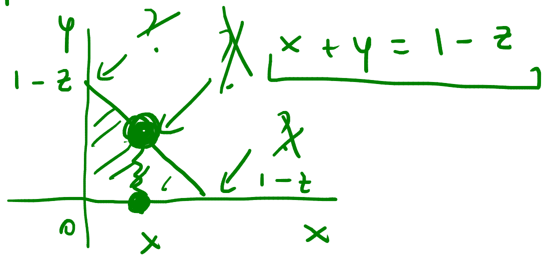
$$= 2\pi \left[r^2 z - \frac{z^3}{3} \right]_0^r = 2\pi \left(r^3 - \frac{r^3}{3} \right) = \frac{4}{3} \pi r^3$$

1. Ricalcolo $\underbrace{\iiint_T (x+z)\,dx\,dy\,dz}_{=:\,I}$



$a = 0, \quad b = 1$

$T_z = ?$



Per strati:

$$I = \int_0^1 \Big(\underbrace{\iint_{T_z} (x+z)\,dx\,dy}_{I_z} \Big)\,dz$$

$$I_z = \int_0^{1-z} \Big(\int_0^{1-z-x} (x+z)\,dy \Big)\,dx$$

$$= \int_0^{1-z} (x+z)(1-z-x)\,dx$$

$$= \int_0^{1-z} \big(x(1-z) - x^2 + z(1-z) - zx \big)\,dx$$

$$= \int_0^{1-z} \big(x(1-2z) - x^2 + z(1-z) \big)\,dx$$

$$= \frac{(1-z)^2}{2}(1-2z) - \frac{(1-z)^3}{3} + z(1-z)^2$$

$$I = \int_0^1 I_z\,dz = \dots$$