30 aprile 2025

Dalla scorsa lezione:

- Es. (sup. cilindrica)

$r > 0$

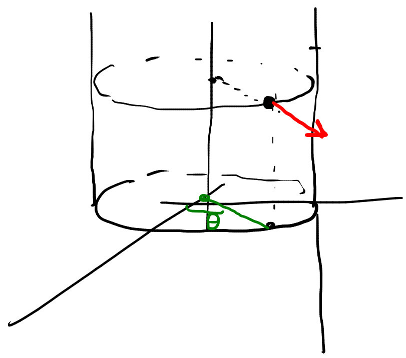
$$\sigma(\theta, z) = (r\cos\theta,\ r\sin\theta,\ z) \qquad (\theta, z) \in \underbrace{[0, 2\pi] \times \mathbb{R}}_{K}$$

$\vdots$

La sup. è regolare; $\forall \theta \in (0, 2\pi) \times \mathbb{R}$:

$$N_\sigma(\theta, z) = (r\cos\theta,\ r\sin\theta,\ 0)$$

$$n_\sigma(\theta, z) = (\cos\theta,\ \sin\theta,\ 0)$$



- Es. (sup. sferica)

$$\sigma(\varphi, \theta) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$$

$$(\varphi, \theta) \in [0, \pi] \times [0, 2\pi]$$

$\vdots$

È una sup. regolare e

$\forall (\varphi, \theta) \in (0, \pi) \times (0, 2\pi)$:

$$n_\sigma(\varphi, \theta) = \frac{N_\sigma(\varphi, \theta)}{\| N_\sigma(\varphi, \theta) \|} = \frac{r\sin\varphi\ \sigma(\varphi, \theta)}{r^2 \sin\varphi}$$

$$= \frac{\sigma(\varphi, \theta)}{r} = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

Es. (sup. regolare con bordo)

Parto dalla sup. cilindrica

$r > 0, \quad h > 0 \qquad K := [0, 2\pi] \times [0, h]$

$$\sigma(\theta, z) = (r\cos\theta, \ r\sin\theta, \ z)$$

C^1 in $\mathbb{R} \times \mathbb{R}$

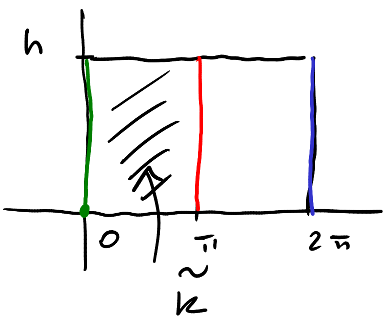
$\forall z$:

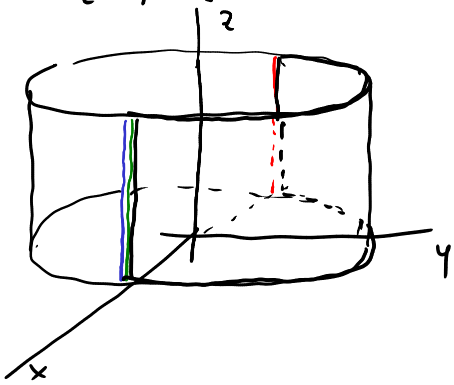
$\sigma(0, z) = \sigma(2\pi, z)$

σ non è ingettiva in K!

$\Rightarrow$ non è una sup. reg. con bordo

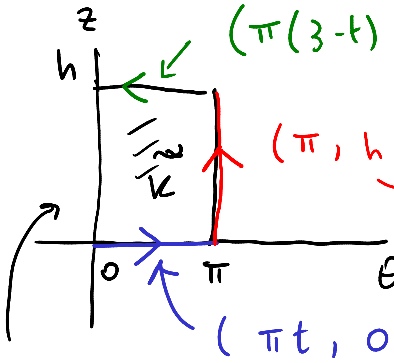
Considero $\tilde{K} := [0, \pi] \times [0, h]$





$\sigma|_{\tilde{K}}$ è ingettiva!

Il "mezzo cilindro" è una sup. reg. con bordo.

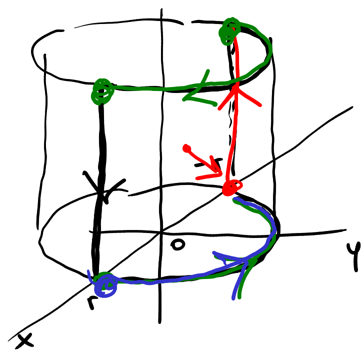


$(\pi(3-t), h) \qquad t \in [2, 3]$

$(\pi, \underbrace{h(t-1)}_{r(t)}) \qquad t \in [1, 2]$

$(\underbrace{\pi t}_{r(t)}, 0) \qquad t \in [0, 1]$

$r(t) = \dots$



$\sigma(r(t)) = \sigma(\pi t, 0) = (r\cos(\pi t), \ r\sin(\pi t), \ 0)$

$\sigma(r(t)) = \sigma(\pi, h(t-1)) = (-r, 0, h(t-1))$

$\sigma(r(t)) = \sigma(\pi(3-t), h) = (r\cos(\pi(3-t)), r\sin(\pi(3-t)), h)$

$\sigma(r(t)) = \ldots$

Es. (sup. grafico)

$f : K \to \mathbb{R}$ di classe C^1

$\sigma : K \to \mathbb{R}^3$ t.c. $\sigma(u,v) = (\underset{e_1}{u}, \underset{e_2}{v}, \underset{e_3}{f(u,v)})$

$$\frac{\partial \sigma}{\partial u}(u,v) = \left(1, 0, \frac{\partial f}{\partial u}(u,v)\right)$$

$$\frac{\partial \sigma}{\partial v}(u,v) = \left(0, 1, \frac{\partial f}{\partial v}(u,v)\right)$$

$$N_\sigma(u,v) = \left(-\frac{\partial f}{\partial u}(u,v), -\frac{\partial f}{\partial v}(u,v), \underbrace{1}_{\neq 0}\right) \neq (0,0,0)$$

$\Rightarrow$ la sup. è regolare!

$$\forall (u,v): \quad n_\sigma(u,v) = \frac{\left(-\frac{\partial f}{\partial u}(u,v), -\frac{\partial f}{\partial v}(u,v), 1\right)}{\sqrt{\underbrace{\left(\frac{\partial f}{\partial u}(u,v)\right)^2 + \left(\frac{\partial f}{\partial v}(u,v)\right)^2}_{= \|\nabla f(u,v)\|_{\mathbb{R}^2}} + 1}}$$

Es. (superficie grafico)

- $f(x,y) = x^2 + y^2 \qquad (x,y) \in \underbrace{\mathbb{R}^2}_{\text{aperto}}$

  $f \in C^1$

  $\Rightarrow \sigma(u,v) := (u, v, u^2 + v^2) \qquad (u,v) \in \mathbb{R}^2$

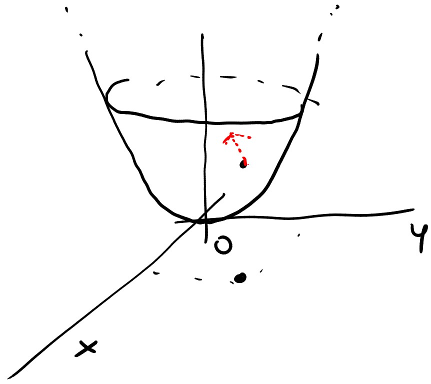
  sup. regolare orientabile

$\forall (u,v) \in \mathbb{R}^2$:

$N_\sigma(u,v) = (-2u, -2v, 1)$

Es: $(u,v) = (1,1)$

$N_\sigma(u,v) = (-2,-2,1)$

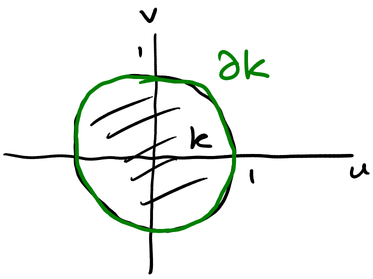


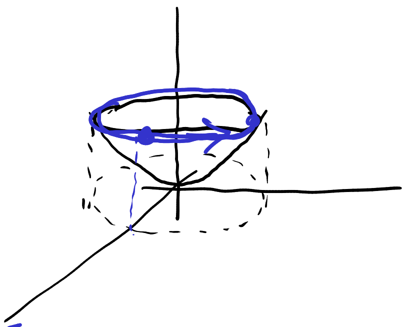
• $f(x,y) = x^2+y^2 \qquad (x,y) \in \bar{B}_1(0,0)$

dom. regolare

$\sigma(u,v) = (u, v, u^2+v^2), \quad (u,v) \in \bar{B}_1(0,0)$

sup. reg. con bordo





$\partial k: r(t) = (\cos t, \sin t)$

$t \in [0, 2\pi]$

$\partial \Sigma: (\sigma \circ r)(t) = \sigma(\cos t, \sin t)$

$= (\cos t, \sin t, \cos^2 t + \sin^2 t)$

$= (\cos t, \sin t, 1)$

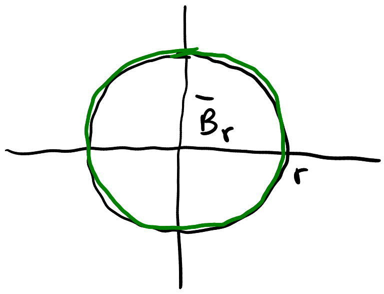
$t \in [0, 2\pi]$

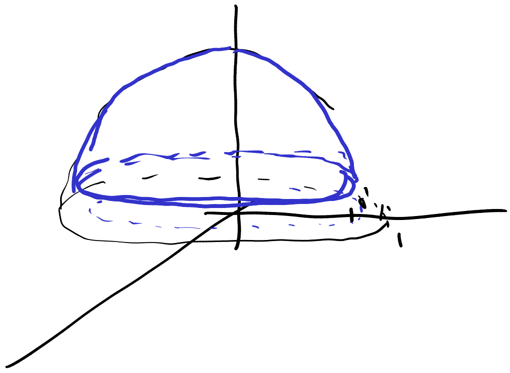
• $f(x,y) = \sqrt{1-x^2-y^2} \qquad (x,y) \in \bar{B}_1(0,0)$

$x^2+y^2 = 1 \Rightarrow 1-x^2-y^2 = 0 \Rightarrow f$ non è deriv. parz.!!

La sup. grafico corrispondente non soddisfa la def. di sup. reg. con bordo!

Se fisso $r \in (0,1)$ e faccio variare (x,y) in $\bar{B}_r(0,0)$, allora f è di classe C^1, quindi la sup. grafico corrispondente è regolare con bordo.





Conveniamo che la "semisfera" sia sup. regolare con bordo uguale alla circonferenza di centro $(0,0,0)$ e raggio 1 nel piano $z=0$.

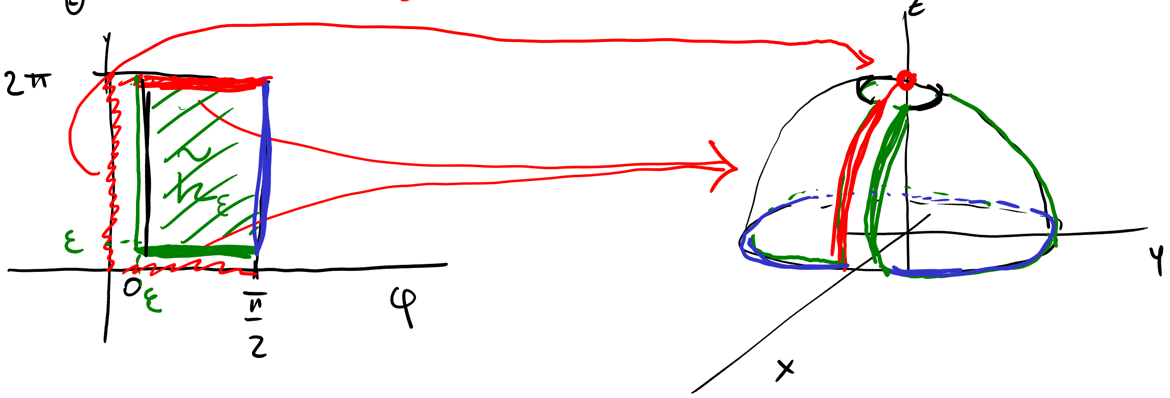
In alternativa: di classe C^1

$$\sigma(\varphi,\theta) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

$$(\varphi,\theta) \in \underbrace{[0,\tfrac{\pi}{2}]\times[0,2\pi]}_{\text{dom. regol.}}$$

Problemi: • perdita di ingettività (per $\varphi=0$ e per $\theta=0$, $\theta=2\pi$)

• $N_\sigma(0,\theta)=0$



$\sigma|_{\tilde{K}_\varepsilon}$ è sup. reg. con bordo

Per $\varepsilon\to 0^+$: ritrovo la "mezza sfera".