17 aprile 2025

Motivo la definizione di polinomio di Taylor

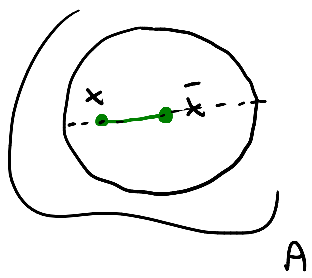
Per ipotesi: $\exists\, r>0$ t.c. $B_r(\bar{x}) \subset A$ e

f è di classe C^2 in $B_r(\bar{x})$

Fisso $x \in B_r(\bar{x})$, $x \neq \bar{x}$, quindi: $\underbrace{\|x-\bar{x}\|_{\mathbb{R}^n}}_{\neq 0} < r$

$$\Rightarrow \frac{r}{\|x-\bar{x}\|_{\mathbb{R}^n}} > 1$$

Definisco $g : \overbrace{\left(-\frac{r}{\|x-\bar{x}\|_{\mathbb{R}^n}}, \frac{r}{\|x-\bar{x}\|_{\mathbb{R}^n}}\right)}^{=:\, I} \to \mathbb{R}$



t.c. $g(t) = f(\underbrace{\bar{x} + t(x-\bar{x})}_{\in B_r(\bar{x})})$

$$\|\bar{x} + t(x-\bar{x}) - \bar{x}\|_{\mathbb{R}^n} = \underbrace{|t|}_{< \frac{r}{\|x-\bar{x}\|_{\mathbb{R}^n}}} \|x-\bar{x}\|_{\mathbb{R}^n}$$

Osservo che $[0,1] \subset I$ e che g è derivabile due volte in I

$\Rightarrow$ posso scrivere il pol. di Taylor di g di centro $t_0 = 0$ e posso valutarlo in $t=1$

$$T_{0,2}(1) = \underbrace{g(0)}_{= f(\bar{x})} + \underbrace{g'(0)}_{= \cancel{?}\ \nabla f(\bar{x}) \cdot (x-\bar{x})} \cdot 1 + \underbrace{\frac{g''(0)}{2}}_{= ?} \cdot 1^2$$

$$g(t) = f(\bar{x} + t(x-\bar{x}))$$

$$\Rightarrow g'(t) = \nabla f(\bar{x} + t(x-\bar{x})) \cdot (x-\bar{x})$$

$$\Rightarrow g'(0) = \nabla f(\bar{x}) \cdot (x-\bar{x})$$

Riscrivo $g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (\bar{x} + t(x-\bar{x}))(x_i - \bar{x}_i)$

$$\Rightarrow g''(t) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} (\bar{x} + t(x-\bar{x})) \right)' (x_i - \bar{x}_i)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i} (\bar{x} + t(x-\bar{x}))(x_j - \bar{x}_j) \right) (x_i - \bar{x}_i)$$

$$\Rightarrow g''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \underbrace{\frac{\partial^2 f}{\partial x_j \partial x_i}}_{= \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x}) \text{ perché } f \in C^2} (\bar{x}) (x_j - \bar{x}_j)(x_i - \bar{x}_i)$$

$$= \dots = H_f(\bar{x})(x-\bar{x}) \cdot (x-\bar{x})$$

Sostituendo in

$$T_{0,2}(1) = g(0) + g'(0) + \frac{1}{2} g''(0)$$

ottengo

$$T_{\bar{x},2}(x) = f(\bar{x}) + \nabla f(\bar{x}) \cdot (x-\bar{x}) + \frac{1}{2} H_f(\bar{x})(x-\bar{x}) \cdot (x-\bar{x})$$

$\square$

Es.

$$f(x,y) = \frac{\cos(x)}{\cos(y)} \qquad (x,y) \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$(0,0)$

$$T_{(0,0),2}(x,y) = f(0,0) + \nabla f(0,0) \cdot (x,y)$$
$$+ \frac{1}{2} H_f(0,0)(x,y) \cdot (x,y)$$

$f(0,0) = 1$

$$\frac{\partial f}{\partial x}(x,y) = -\frac{\sin(x)}{\cos(y)} \quad , \quad \frac{\partial f}{\partial y}(x,y) = \cos(x)\left(-\frac{-\sin(y)}{\cos^2(y)}\right)$$

$$= \cos(x)\frac{\sin(y)}{\cos^2(y)}$$

$$\nabla f(0,0) = (0,0)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{\cos(x)}{\cos(y)}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -\sin(x)\frac{\sin(y)}{\cos^2(y)}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \cos(x)\frac{\cos^3(y) - \sin(y)\,2\cos(y)\,(-\sin(y))}{\cos^4(y)}$$

$$H_f(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T_{(0,0),2}(x,y) = 1 + \underbrace{(0,0)\cdot(x,y)}_{=0} + \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + \frac{1}{2}\begin{pmatrix} -x+0 \\ 0+y \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}$$

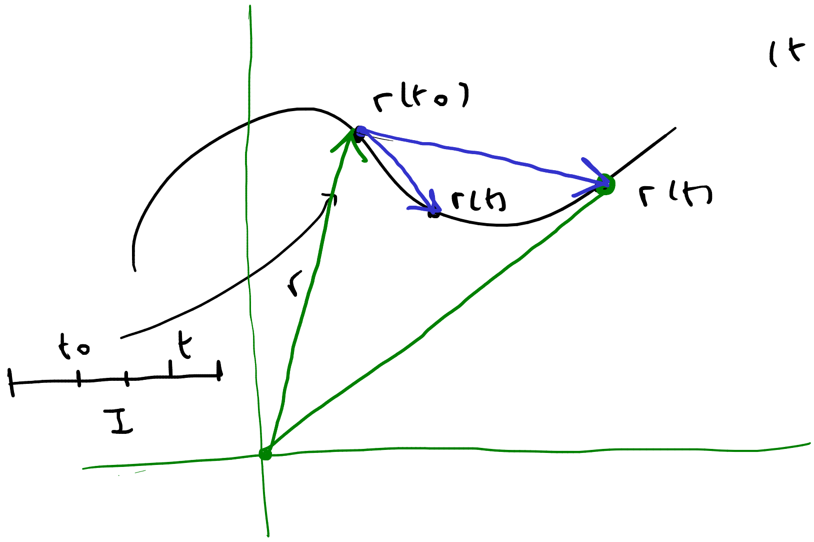
$$= 1 + \frac{1}{2}\begin{pmatrix} -x \\ y \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + \frac{1}{2}(-x^2+y^2) = 1 - \frac{x^2}{2} + \frac{y^2}{2}$$

Per $(x,y) \to (0,0)$:

$$\frac{\cos(x)}{\cos(y)} = 1 - \frac{x^2}{2} + \frac{y^2}{2} + o(x^2+y^2)$$

$\square$



$(t \neq t_0)$

$$\frac{r(t) - r(t_0)}{t - t_0}$$

$$r'(t_0) \overset{\text{def}}{=} \lim_{t \to t_0} \frac{r(t) - r(t_0)}{t - t_0}$$

Es. (curve regolari)

- $x, y \in \mathbb{R}^n$, $x \neq y$

$r : [0,1] \to \mathbb{R}^n$ $\qquad r(t) = x + t(y - x)$

$\forall i$: $r_i(t) = x_i + t(y_i - x_i)$ di classe C^1

$\Rightarrow$ r di classe C^1

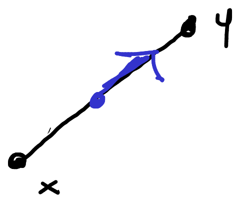
$r(0) = x$, $r(1) = y$, $x \neq y$

curva non chiusa

$\forall t$: $r'(t) = y - x \neq 0$ $\qquad (x \neq y)$

Quindi : curva regolare

Versore tangente in $t_0 \in [0,1]$:



$$T(t_0) = \frac{r'(t_0)}{\|r'(t_0)\|_{\mathbb{R}^n}} = \frac{y - x}{\|y - x\|_{\mathbb{R}^n}}$$

- $r : [0, 2\pi] \to \mathbb{R}^2$ t.c. $r(t) = (\cos(t), \sin(t))$

di classe C^1

r di classe C^1 ✓

$r(0) = (1,0) = r(2\pi)$ curva chiusa

$\forall t \in [0, 2\pi]$: $r'(t) = (-\sin(t), \cos(t))$

$r'(0) = (0,1)$
$\quad\quad\quad\quad \shortparallel$ ✓
$r'(2\pi) = (0,1)$

$r'(t) \neq (0,0) \quad \forall t \in [0, 2\pi]$ ~~?~~ ✓

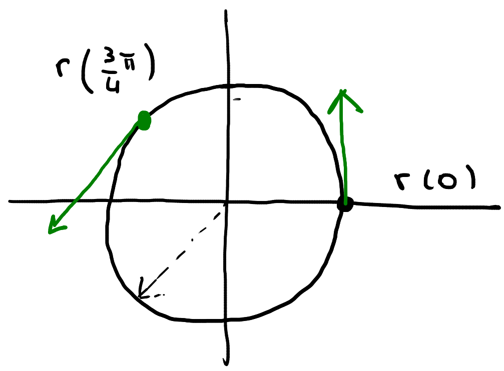
$\Updownarrow$

$\|r'(t)\| \neq 0 \iff \|r'(t)\| > 0 \iff \|r'(t)\|^2 > 0$

$\forall t \in [0, 2\pi]$: $\|r'(t)\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = \underbrace{1}_{\neq 0}$

Quindi: curva regolare

$\forall t \in [0, 2\pi]$: $T(t) = \dfrac{r'(t)}{\underbrace{\|r'(t)\|}_{=1}} = (-\sin(t), \cos(t))$



$T(0) = (0,1)$

$T\left(\frac{3}{4}\pi\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

- $r : [-1,1] \to \mathbb{R}^2$ t.c. $r(t) = (t^3, t^2)$

classe C^1 ✓

$r(-1) = (-1, 1)$
$\quad\quad\quad\quad\quad \neq$ curva non chiusa
$r(1) = (1,1)$

$\forall t \in [-1,1]$: $r'(t) = (3t^2, 2t)$

$r'(t) = (0,0) \iff t = 0$!!

La curva non è regolare, ma è regolare a tratti:

$$[-1,1] = [-1,0] \cup [0,1]$$

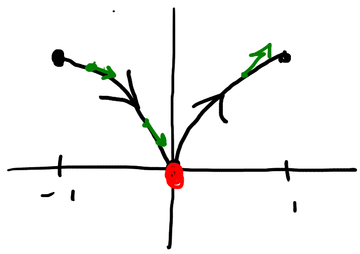
- $r|_{[-1,0]}$, $r|_{[0,1]}$ di classe C^1
- $\forall t \in (-1,0) \cup (0,1)$: $r'(t) \neq (0,0)$

$$T(t) = \frac{(3t^2, 2t)}{\sqrt{9t^4 + 4t^2}}$$

Disegno il sostegno della curva.

Parto dalle eq. parametriche:

$$\begin{cases} x = t^3 \\ y = t^2 \end{cases} \rightarrow t = \sqrt[3]{x} \qquad y = \sqrt[3]{x^2} \quad x \in [-1,1]$$



- $r : \mathbb{R} \to \mathbb{R}^3$ t.c. $r(t) = (\underset{C^1}{a\cos(t)}, \underset{C^1}{b\sin(t)}, \underset{C^1}{ct})$

$(a, b > 0, \ c \neq 0)$

$\Rightarrow r$ di classe C^1 ✓

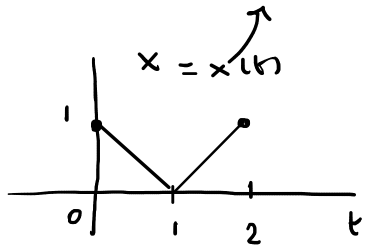
$I = \mathbb{R}$ non curva chiusa ✓

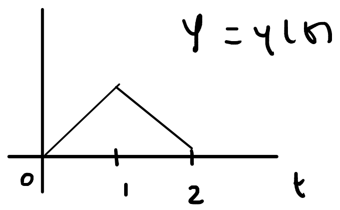
$\forall t \in \mathbb{R}$: $r'(t) = (-a\sin(t), b\cos(t), \underbrace{c}_{\neq 0}) \neq (0,0,0)$

Quindi: curva regolare

Esempi (curve regolari a tratti)

- $r(t) = (|t-1|, \ 1 - |t-1|) \qquad t \in [0,2]$





$r|_{[0,1]}$, $r|_{[1,2]}$ di classe C^1

$\forall t \in (0,1) \cup (1,2)$:

$$r'(t) = \Big(\text{sign}(t-1), \ - \text{sign}(t-1) \Big) \neq (0,0)$$

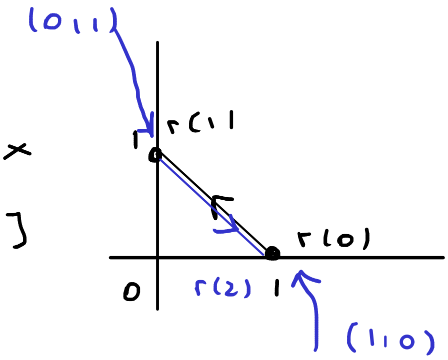
Quindi: la curva è regolare a tratti.

Sostegno ?

$x = |t-1|$

$y = 1 - |t-1|$ $\Rightarrow$ $y = 1 - x$

$x \in [0,1]$



- $r(t) = (\cos^3(t), \ \sin^3(t)) \qquad t \in [0, 2\pi]$

r di classe C^1 ✓

$r(0) = (1,0) = r(2\pi)$ curva chiusa

$\forall t \in [0, 2\pi]$: $r'(t) = \Big(\underbrace{3\cos^2(t)(-\sin(t))}_{\sim -3t}, \ 3\underbrace{\sin^2(t)}_{3t^2}\cos(t) \Big)$

$t \to 0$

$r'(0) = (0,0)$!!! curva non regolare

$\forall t$: $\|r'(t)\|^2 = 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)$

$$= 9\cos^2(t)\sin^2(t)\,(\cos^2(t)+\sin^2(t))$$

$$= 9\cos^2(t)\sin^2(t)$$

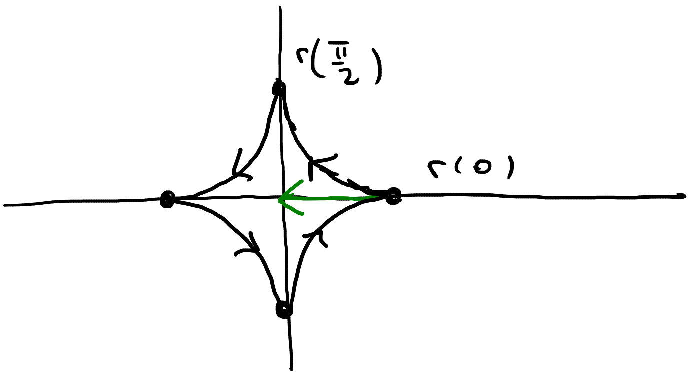
$\|r'(t)\| = 0 \iff \cos(t) = 0$ oppure $\sin(t) = 0$

$$\iff t \in \left\{0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi\right\}$$

$$[0,2\pi] = \left[0,\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2},\pi\right] \cup \left[\pi,\frac{3}{2}\pi\right] \cup \left[\frac{3}{2}\pi, 2\pi\right]$$

nei punti interni: $r'(t) \neq (0,0)$

Quindi: la curva è regolare a tratti



Aggiungo qualche osservazione sull'andamento per $t \to 0$:

$r'(t) = (-3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t))$

Per $t \notin \left\{0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi, 2\pi\right\}$:

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{(-3\cos^2(t)\sin(t),\ 3\sin^2(t)\cos(t))}{3\,|\cos(t)|\,|\sin(t)|}$$

$$= \left(-|\cos(t)|\,\operatorname{sign}(\sin(t)),\ \operatorname{sign}(\cos(t))\,|\sin(t)|\right)$$

Per $t \to 0^+$: $T(t) = (\underbrace{-\cos(t)}_{\sim -1},\ \underbrace{\sin(t)}_{\sim 0})$