15 aprile 2025

Es. $f: \mathbb{R}^2 \to \mathbb{R}^3$ t.c. $f(x,y) = \begin{pmatrix} x+y \\ xy \\ xy^2 \end{pmatrix}$ polin. $\Rightarrow$ diff. ✓

$g: \mathbb{R}^3 \to \mathbb{R}$ t.c. $g(u,v,w) = \underbrace{uv^2w}_{\text{polinom.} \Rightarrow \text{diff.} ✓}$

Calcolo $g \circ f : \mathbb{R}^2 \to \mathbb{R}$

$\forall (x,y) \in \mathbb{R}^2$: $(g\circ f)(x,y) = g(f(x,y))$

$= g(\overset{u}{x+y}, \overset{v}{xy}, \overset{w}{xy^2}) = (x+y)(xy)^2(xy^2)$

$= (x+y)\, x^3y^4 = x^4y^4 + x^3y^5$

$J_{g\circ f}(x,y) = \nabla(g\circ f)(x,y) = (4x^3y^4 + 3x^2y^5 \quad 4x^4y^3 + 5x^3y^4)$

Calcolo separatamente le matrici jacobiane di f e g.

$J_g(u,v,w) = \nabla g(u,v,w) = (v^2w \quad 2uvw \quad uv^2)$

$J_g(f(x,y)) = J_g(\overset{u}{x+y}, \overset{v}{xy}, \overset{w}{xy^2})$

$= ((xy)^2(xy^2) \quad 2(x+y)(xy)(xy^2) \quad (x+y)(xy)^2)$

$= (x^3y^4 \quad 2(x+y)x^2y^3 \quad (x+y)x^2y^2)$

$J_f(x,y) = \begin{pmatrix} 1 & 1 \\ y & x \\ y^2 & 2xy \end{pmatrix}$ $f(x,y) = \begin{pmatrix} x+y \\ xy \\ xy^2 \end{pmatrix}$

$\Rightarrow J_g(f(x,y))\, J_f(x,y) =$

$= (x^3y^4 \quad 2(x+y)x^2y^3 \quad (x+y)x^2y^2) \begin{pmatrix} 1 & 1 \\ y & x \\ y^2 & 2xy \end{pmatrix}$

$$= \Big(x^3y^4 + 2(x+y)\,x^2y^3 \cdot y + (x+y)\,x^2y^2\,y^2 \qquad x^3y^4 + 2(x+y)\,x^2y^3\,x + (x+y)\,x^2y^2\,2xy \Big)$$

$$= \Big(x^3y^4 + 2x^3y^4 + 2x^2y^5 + x^3y^4 + x^2y^5 \qquad \underline{x^3y^4} + 2x^4y^3 + 2\underline{x^3y^4} + 2x^4y^3 + 2\underline{x^3y^4} \Big)$$

$$= \Big(4x^3y^4 + 3x^2y^5 \qquad \underline{5x^3y^4} + 4x^4y^3 \Big)$$

- $f(x) = \begin{pmatrix} x+2 \\ x^2+x \end{pmatrix}$ $\qquad f: \mathbb{R} \to \mathbb{R}^2$, classe C^1

$g(u,v) = u^2 - 2v^3$ $\qquad g: \mathbb{R}^2 \to \mathbb{R}$, classe C^1

$g \circ f : \mathbb{R} \to \mathbb{R}$

$$(g\circ f)(x) = g(f(x)) = g(x+2,\ x^2+x) = (x+2)^2 - 2(x^2+x)^3$$

$$(g\circ f)'(x) = 2(x+2) - 6(x^2+x)^2(2x+1) = \dots$$

$$\nabla g(u,v) = (2u,\ -6v^2)$$

$$\nabla g(f(x)) \cdot f'(x) = \big(2(x+2),\ -6(x^2+x)^2\big) \cdot (1,\ 2x+1)$$

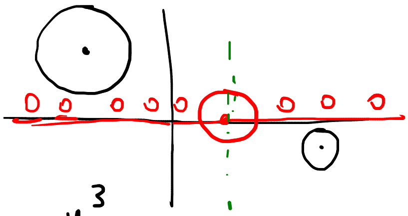
$$= 2(x+2)\cdot 1 - 6(x^2+x)^2(2x+1)$$

Es. (sull'ordine di derivazione)

$$f(x,y) = \begin{cases} y^2 \arctan\left(\dfrac{x}{y}\right) & (x,y) \in \mathbb{R} \times \mathbb{R}^* \\ 0 & (x,y) \in \mathbb{R} \times \{0\} \end{cases}$$

Calcolo le derivate seconde miste in $(0,0)$

Calcolo $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$



$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y^2 \dfrac{1}{1+\frac{x^2}{y^2}} \cdot \dfrac{1}{y} = \dfrac{y^3}{x^2+y^2} & y \neq 0 \\[2ex] \displaystyle\lim_{t\to 0} \frac{\overset{=0}{f(x+t,0)} - \overset{=0}{f(x,0)}}{t} = 0 & y = 0 \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) \overset{\cancel{?}}{=} \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(0,t) - \frac{\partial f}{\partial x}(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^3}{0^2+t^2} - 0}{t} = \lim_{t\to 0} \frac{t}{t} = 1$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} 2y \arctan\left(\frac{x}{y}\right) + y^2 \dfrac{1}{1+\frac{x^2}{y^2}} \left(-\dfrac{x}{y^2}\right) & y \neq 0 \\[1ex] = 2y \arctan\left(\frac{x}{y}\right) - \dfrac{xy^2}{x^2+y^2} & \\[2ex] \displaystyle\lim_{t\to 0} \frac{f(x,t) - f(x,0)}{t} & y = 0 \end{cases}$$

$$= \lim_{t\to 0} \frac{t^2 \arctan\left(\frac{x}{t}\right) - 0}{t}$$

$$= \lim_{t\to 0} \underbrace{t}_{\downarrow\, 0} \underbrace{\left(\arctan\left(\frac{x}{t}\right)\right)}_{\text{limitata}} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \overset{\cancel{?}}{=} \lim_{t\to 0} \frac{\overset{=0}{\frac{\partial f}{\partial y}(t,0)} - \overset{=0}{\frac{\partial f}{\partial y}(0,0)}}{t} = 0$$

Conclusione: $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 1 \neq 0 = \dfrac{\partial^2 f}{\partial x \partial y}(0,0)$

Es. (sulla matrice hessiana)

$f: \mathbb{R}^3 \to \mathbb{R}$ $\qquad f(x,y,z) = x^5 + y^4 z^3 - 3xz^2$

polinomiale $\Rightarrow$ di classe C^2

$\forall (x,y,z) \in \mathbb{R}^3$:

$\frac{\partial f}{\partial x}(x,y,z) = 5x^4 - 3z^2 \qquad \frac{\partial f}{\partial y}(x,y,z) = 4y^3 z^3$

$\frac{\partial f}{\partial z}(x,y,z) = 3y^4 z^2 - 6xz$

$$H_f(x,y,z) = \begin{pmatrix} 20x^3 & 0 & -6z \\ 0 & 12y^2 z^3 & 12y^3 z^2 \\ -6z & 12y^3 z^2 & 6y^4 z - 6x \end{pmatrix}$$

~~OSS: $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^2 \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$~~

Commenti sul teorema del valor medio

~~.... $\exists z \in [x,y] \setminus \{x,y\}$ t.c.~~

~~$f_1(y) - f_1(x) = \nabla f_1(z) \cdot (y-x)$~~

~~$= ??$~~

~~$\exists ? \in [x,y] \setminus \{x,y\}$ t.c.~~

~~$f_2(y) - f_2(x) = \nabla f_2(?) \cdot (y-x)$~~

Es: $f: [0, 2\pi] \to \mathbb{R}^2 \qquad f(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$f(2\pi) - f(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

? $\exists t_0 \in (0, 2\pi)$ t.c. $\begin{pmatrix} \cos'(t_0) \\ \sin'(t_0) \end{pmatrix} 2\pi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \underline{\underline{\text{No!!}}}$