10 aprile 2025

Es:

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Già noto: $\nexists \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$

$\Rightarrow$ f non continua in $(0,0)$

$\Rightarrow$ f non differenziabile in $(0,0)$

Facile: $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0 \Rightarrow \nabla f(0,0) = (0,0)$

$\Rightarrow \nabla f(0,0) \cdot v = 0 \quad \forall v \in \mathbb{R}^n$

$v = (v_1, v_2)$ con $v_1, v_2 \neq 0$; $\forall t \neq 0$:

$$\frac{f((0,0)+t(v_1,v_2)) - f(0,0)}{t} = \frac{f(tv_1, tv_2) - f(0,0)}{t}$$

$$= \frac{\dfrac{t^2 v_1^2 \, t v_2}{t^4 v_1^4 + t^2 v_2^2} - 0}{t} = \frac{t^3 v_1^2 v_2}{t^4 v_1^4 + t^2 v_2^2} \cdot \frac{1}{t}$$

$$= \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} \xrightarrow[t\to 0]{} \underbrace{\frac{v_1^2}{v_2}}_{\neq 0} =: \frac{\partial f}{\partial v}(0,0)$$

Per $f(x,y) = \begin{cases} \left(\dfrac{x^2 y}{x^4+y^2}\right)^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

... tutto uguale tranne :

$$\frac{f(tv_1, tv_2) - f(0,0)}{t} = \frac{\left(\frac{t^3 v_1^2 v_2}{t^4 v_1^4 + t^2 v_2^2}\right)^2 - 0}{t}$$

$$\frac{\cancel{t^2}\, v_1^4 v_2^2}{(t^2 v_1^4 + v_2^2)^2} \cdot \frac{1}{\cancel{t}} \xrightarrow[t \to 0]{} 0 = \frac{\partial f}{\partial v}(0,0)$$

$$\| \quad \nabla f(0,0) \cdot v$$

Oss: f diff. in $\bar{x}$, $\nabla f(\bar{x}) \neq 0$

$$v := \pm \frac{\nabla f(\bar{x})}{\|\nabla f(\bar{x})\|_{\mathbb{R}^n}} \quad : \quad \|v\|_{\mathbb{R}^n} = 1$$

$$\frac{\partial f}{\partial v}(\bar{x}) = \nabla f(\bar{x}) \cdot (\pm v) = \nabla f(\bar{x}) \cdot \frac{\pm \nabla f(\bar{x})}{\|\nabla f(\bar{x})\|_{\mathbb{R}^n}}$$

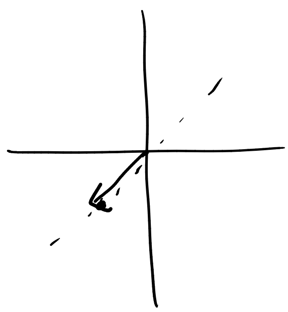
$$= \frac{\pm 1}{\|\nabla f(\bar{x})\|_{\mathbb{R}^n}} \nabla f(\bar{x}) \cdot \nabla f(\bar{x})$$

$$= \frac{\pm 1}{\cancel{\|\nabla f(\bar{x})\|_{\mathbb{R}^n}}} \|\nabla f(\bar{x})\|^{\cancel{2}}_{\mathbb{R}^n} = \pm \|\nabla f(\bar{x})\|_{\mathbb{R}^n}$$

$$f(x,y) = 4 - x^2 - y^2 \qquad (1,1)$$

$$\frac{\partial f}{\partial x}(x,y) = -2x \quad , \quad \frac{\partial f}{\partial y}(x,y) = -2y$$

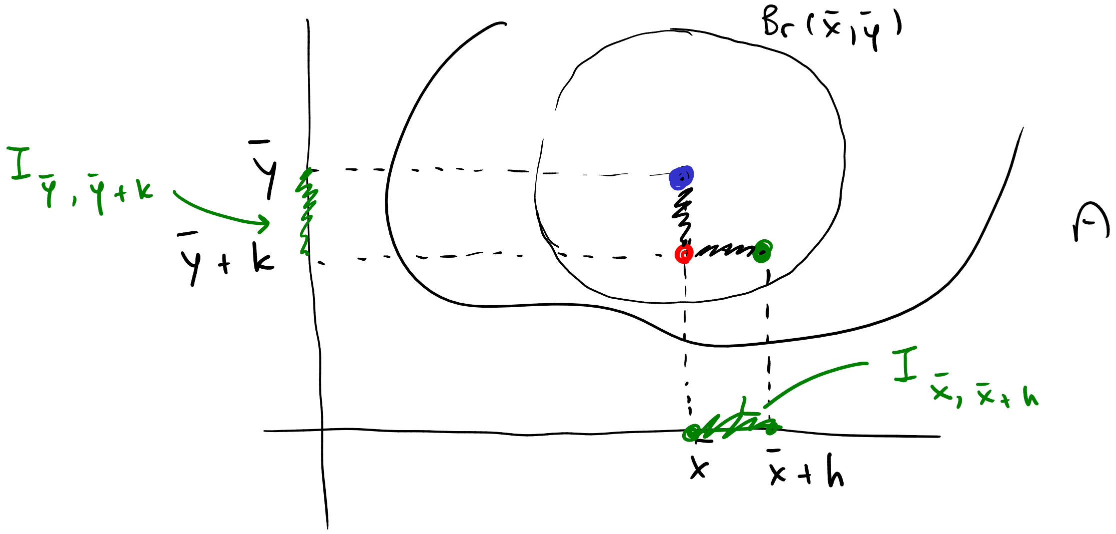
$$\nabla f(1,1) = (-2, -2)$$



$$\|\nabla f(1,1)\| = \sqrt{4+4} = 2\sqrt{2}$$

$$\frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Nella dim. del TDT:



<u>Es</u>.

- $f(x,y) = \sqrt{x^2+y^2} \qquad (x,y) \in \mathbb{R}^2$

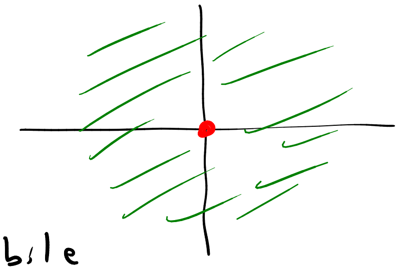
In $(0,0)$: non diff. (già detto)

In $\mathbb{R}^2 \setminus \{(0,0)\}$ (aperto)

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

cont. cont. cont. cont.

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}} \quad \text{cont.}$$



$\Rightarrow$ In $\mathbb{R}^2 \setminus \{(0,0)\}$ f è differenziabile