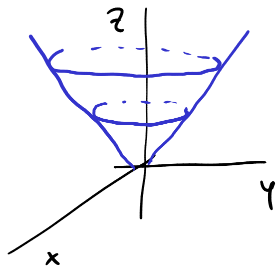
4 aprile 2025

Es.



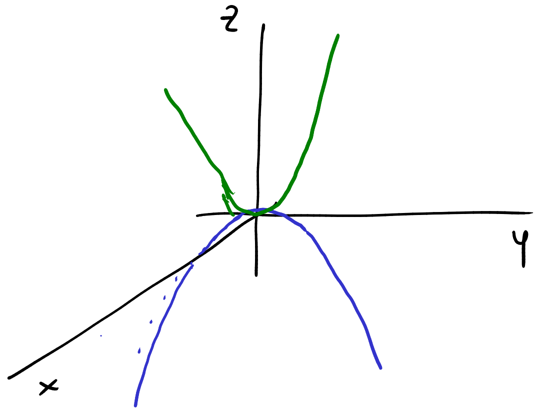
- $f(x,y) = \sqrt{x^2+y^2} \qquad (x,y) \in \mathbb{R}^2$

Parametrizzazione della sup. grafico corrispondente:

$\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ t.c. $\sigma(u,v) = (u, v, \sqrt{u^2+v^2})$

- $f(x,y) = x^2 - y^2 \qquad (x,y) \in \mathbb{R}^2$

$\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ t.c. $\sigma(u,v) = (u, v, u^2 - v^2)$



- $r > 0 \qquad f(x,y) = \sqrt{r^2 - x^2 - y^2}$

$r^2 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq r^2$

$(x,y) \in \bar{B}_r(0,0)$

$\sigma : \bar{B}_r(0,0) \to \mathbb{R}^3, \quad \sigma(u,v) = (u, v, \sqrt{r^2 - u^2 - v^2})$

$(x,y,z) \in \text{graf}(f) \iff (x,y) \in \bar{B}_r(0,0)$ e $z = f(x,y)$

$\Updownarrow$

$(\geq 0)\ z = \sqrt{r^2 - x^2 - y^2}$

$\Updownarrow$

$z^2 = r^2 - x^2 - y^2$

$\Updownarrow$

(x,y,z) è nella metà superiore della sfera di centro $(0,0,0)$ e raggio r $\iff x^2 + y^2 + z^2 = r^2$

$- (0,0,0)$

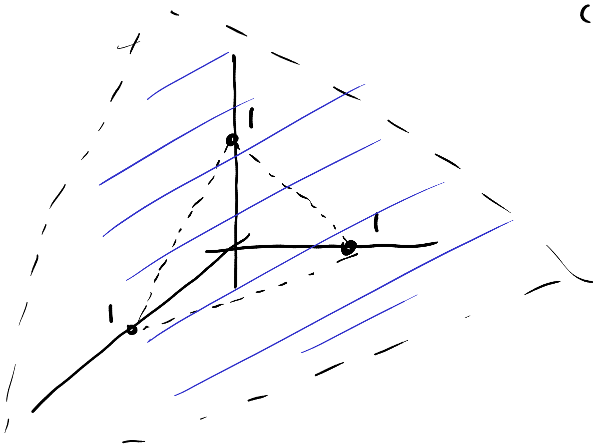
$\|(x,y,z)\| = r$

$f(x,y) = 1 - x - y \qquad (x,y) \in \mathbb{R}^2$

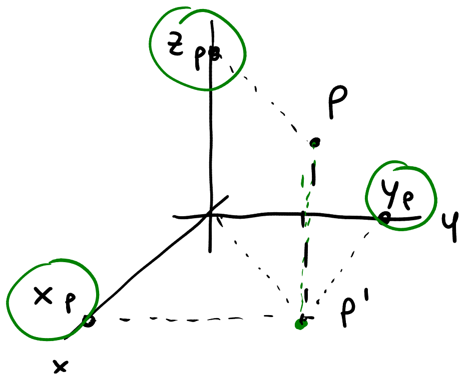
$\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ t.c. $\sigma(u,v) = (u, v, 1-u-v)$

$(x,y,z) \in \text{graf}(f) \iff z = 1 - x - y$

$\iff x + y + z - 1 = 0$ equazione di un piano

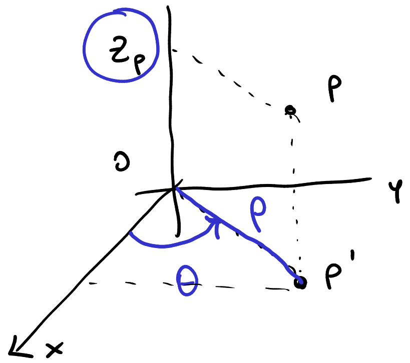


Coordinate cilindriche in $\mathbb{R}^3$



(x, y, z)

coord. cartesiane



(ρ, θ, z) coord. cilindriche

"Traduzione" ?

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$\rho \in [0, +\infty)$

$\theta \in [0, 2\pi]$ (oppure $[-\pi, \pi]$)

$z \in \mathbb{R}$

x = costante $\iff$ piano parallelo al piano yz

y = costante $\iff$ ...... xz

z = costante $\iff$ piano orizzontale

$\rho =$ costante $\longleftrightarrow$ cilindro ...

$\theta =$ costante $\longleftrightarrow$ semipiano

$z =$ costante $\longleftrightarrow$ come sopra

Oss. sulla superficie cilindrica

$\sigma : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ t.c. $\sigma(\theta, z) = (r\cos(\theta), r\sin(\theta), z)$

$(\theta_1, z_1), (\theta_2, z_2) \in [0, 2\pi] \times \mathbb{R}$

$(\theta_1, z_1) \neq (\theta_2, z_2)$ e $(\underline{\theta_1}, z_1) \in \underline{(0, 2\pi)} \times \mathbb{R}$

Può essere $\underset{(*)}{\sigma(\theta_1, z_1) = \sigma(\theta_2, z_2)}$?

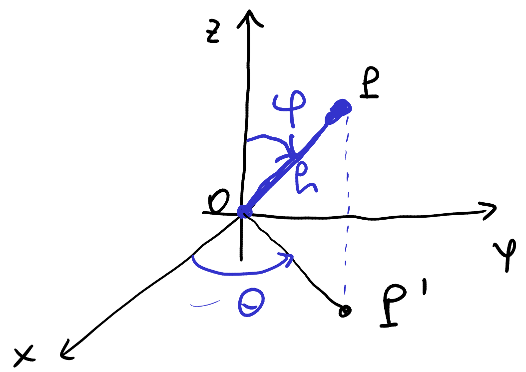
$(*) \iff (r\cos(\theta_1), r\sin(\theta_1), z_1) = (r\cos(\theta_2), r\sin(\theta_2), z_2)$

$$\iff \begin{cases} \cancel{r}\cos(\theta_1) = \cancel{r}\cos(\theta_2) \\ \cancel{r}\sin(\theta_1) = \cancel{r}\sin(\theta_2) \\ z_1 = z_2 \end{cases} \overset{\theta_1 \notin \{0, 2\pi\}}{\Longrightarrow} \theta_1 = \theta_2$$

L'unica possibilità affinché valga $(*)$ è che sia $(\theta_1, z_1) = (\theta_2, z_2)$.

Quindi: la condizione di "semplicità" è verificata.

Coord. sferiche (polari) in $\mathbb{R}^3$



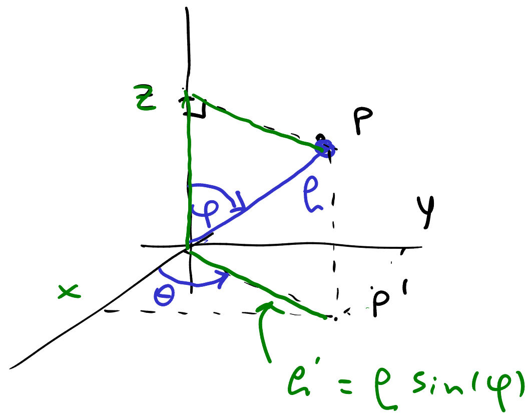
co-latitudine

(ρ, φ, θ)

$\rho \in [0, +\infty)$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi]$

"Traduzione"?

$$\begin{cases} x = \rho \sin(\varphi)\cos(\theta) \\ y = \rho \sin(\varphi)\sin(\theta) \\ z = \rho \cos(\varphi) \end{cases}$$



ρ = costante $\leftrightarrow$ sfera di centro $(0,0,0)$

φ = costante $\leftrightarrow$ cono

θ = costante $\leftrightarrow$ semipiano

Es.

- $f(x_1, x_2) = \frac{x_1^2}{2} + x_2^2 \qquad (x_1, x_2) \in \mathbb{R}^2$

$\bar{x} = (3,1) \qquad v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

punto interno $\qquad \|v\|_{\mathbb{R}^2} = 1$

? $\lim_{t \to 0} \frac{f(\bar{x}+tv) - f(\bar{x})}{t} = \lim_{t\to 0} \frac{f\left((3,1) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) - f(3,1)}{t}$

$$= \lim_{t\to 0} \frac{f\left(3 + t \cdot \frac{1}{\sqrt{2}}, 1 + t \cdot \frac{1}{\sqrt{2}}\right) - f(3,1)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{1}{2}\left(3 + \frac{t}{\sqrt{2}}\right)^2 + \left(1 + \frac{t}{\sqrt{2}}\right)^2 - \left(\frac{9}{2} + 1\right)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{1}{2}\left(\cancel{9} + \frac{t^2}{2} + \frac{6}{\sqrt{2}}t\right) + \cancel{1} + \frac{t^2}{2} + \frac{2}{\sqrt{2}}t - \left(\cancel{\frac{9}{2}} + \cancel{1}\right)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^2}{4} + \frac{3}{\sqrt{2}}t + \frac{t^2}{2} + \frac{2}{\sqrt{2}}t}{t} = \lim_{t\to 0} \frac{\left(\frac{t}{4}+\frac{t}{2}\right) + \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}}}{1}$$

$$= \frac{5}{\sqrt{2}} \in \mathbb{R}$$

$\Rightarrow$ f è derivabile in $(3,1)$ nella direzione $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

e $\quad \frac{\partial f}{\partial v}(3,1) = \frac{5}{\sqrt{2}}$

In alternativa, definisco $g : \mathbb{R} \to \mathbb{R}$ t.c.

$$g(t) := f(\bar{x} + tv) = \dots = \frac{1}{2}\left(3 + \frac{t}{\sqrt{2}}\right)^2 + \left(1 + \frac{t}{\sqrt{2}}\right)^2$$

Oss: g è funz. polinomiale, quindi derivabile in $\mathbb{R}$; in particolare, g è derivabile in $t = 0$,

quindi: f è der. in $(3,1)$ nella direzione v.

Inoltre:

$$\forall t \in \mathbb{R} : \quad g'(t) = \frac{1}{\cancel{2}} \cdot \cancel{2}\left(3 + \frac{t}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} + 2\left(1 + \frac{t}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{\partial f}{\partial v}(3,1) = g'(0) = 3 \cdot \frac{1}{\sqrt{2}} + 2 \cdot 1 \cdot \frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}}.$$