1 aprile 2025

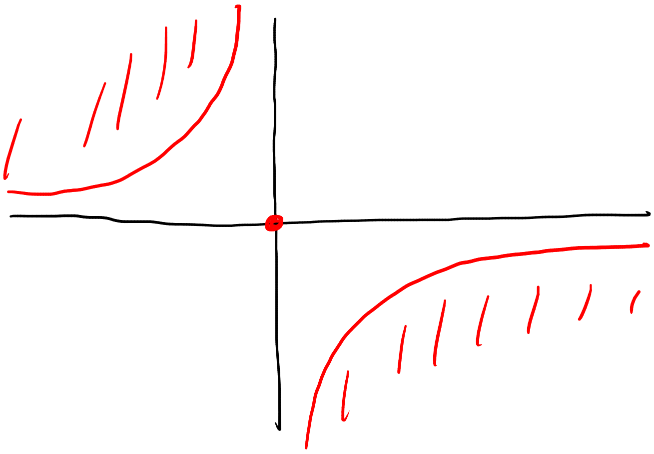
Es.

- $f(x,y) = \dfrac{\ln(1+xy)}{x^2+y^2}$

$1+xy>0 \qquad xy>-1$

$x^2+y^2\neq 0 \qquad (x,y)\neq(0,0)$

$\mathbb{R}^2 \setminus \mathrm{dom}(f)$



$\mathrm{dom}(f) = \{(x,y) \mid (x,y)\neq(0,0),$
$xy>-1\}$

↑

aperto, illimitato

f continua $\Rightarrow$ limiti significativi:

- $(x,y)\to(0,0)$
- $(x,y)\to\left(a,-\frac{1}{a}\right) \qquad a\neq 0$
- $\|(x,y)\|\to+\infty$

Fisso $a\neq 0$

$$\lim_{(x,y)\to(a,-\frac{1}{a})} f(x,y) = \lim_{(x,y)\to(a,-\frac{1}{a})} \frac{\ln(1+\overbrace{xy}^{\to a(-\frac{1}{a})})}{\underbrace{x^2+y^2}_{\to a^2+\frac{1}{a^2}>0}} = -\infty$$

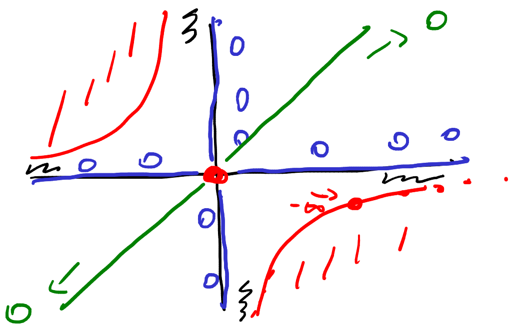
$-\infty \qquad 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\ln(1+\overbrace{xy}^{\to 0})}{\underbrace{x^2+y^2}_{\to 0^+}} \qquad \frac{0}{0} \quad !!$$

$0 \qquad \to 1$

Oss: $f|_{\text{asse } x} \equiv f|_{\text{asse } y} \equiv 0$

$\Rightarrow$ se esiste, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$



$B := \{ (x,y) \in \mathrm{dom}(f) \mid x = y \}$

$$F_{|_B}(x,y) = f(x,x) = \frac{\ln(1+x\cdot x)}{x^2+x^2} = \frac{\ln(1+x^2)}{2x^2}$$

$$\lim_{(x,y)\to(0,0)} F_{|_B}(x,y) = \lim_{x\to 0} \overbrace{\frac{\ln(1+x^2)}{2x^2}}^{\to 1} = \frac{1}{2} \neq 0$$

$t \to 0 : \quad \frac{\ln(1+t)}{t} \to 1 \quad \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$

Studio $\lim_{\|(x,y)\|\to+\infty} f(x,y)$

Oss: se esiste, deve essere uguale a 0.

Provo:

$$\lim_{\|(x,y)\|\to+\infty} F_{|_B}(x,y) = \lim_{|x|\to+\infty} \frac{\ln(\overbrace{1+x^2}^{\to+\infty})}{2\underbrace{x^2}_{\to+\infty}}$$

$$= \frac{1}{2} \lim_{t\to+\infty} \frac{\ln(1+t)}{t} = 0$$

gerarchia di infiniti

conferma la congettura che il limite sia 0

Verifico che, posto $C := \{ (x,y) \in \mathrm{dom}(f) \mid xy > 0 \}$,

risulta

$$\lim_{\|(x,y)\|\to+\infty} F_{|_C}(x,y) = 0$$

$\forall (x,y) \in C$:

$$0 < xy = |xy| \le \frac{x^2+y^2}{2}$$

$$\Rightarrow 1 + xy \le 1 + \frac{x^2+y^2}{2}$$

ln crescente

$$\Rightarrow \quad \ln(1+xy) \le \ln\left(1 + \frac{x^2+y^2}{2}\right)$$

$x^2+y^2>0$

$$\Rightarrow \quad \frac{\ln(1+xy)}{x^2+y^2} \le \frac{\ln\left(1+\frac{x^2+y^2}{2}\right)}{x^2+y^2}$$

Quindi:

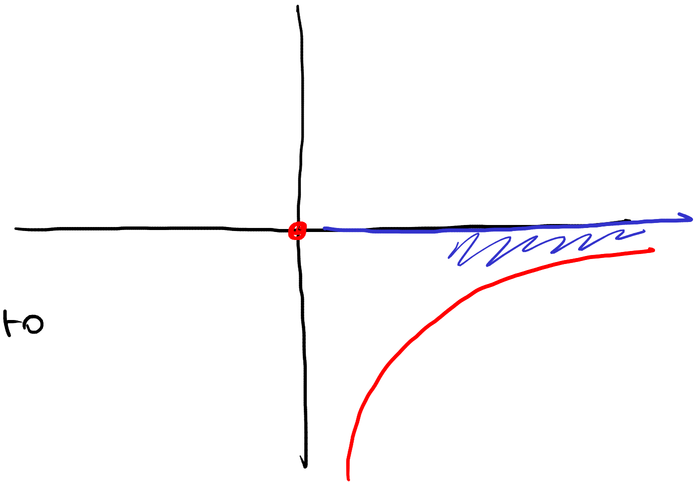
$$\forall (x,y) \in \underline{C} \qquad 0 \le f(x,y) \le \frac{\ln\left(1+\frac{x^2+y^2}{2}\right)}{x^2+y^2}$$

$$\lim_{\substack{\|(x,y)\|\to+\infty \\ \| \\ \sqrt{x^2+y^2}}} \frac{\ln\left(1+\frac{x^2+y^2}{2}\right)}{x^2+y^2} = \lim_{\substack{t\to+\infty \\ t=x^2+y^2}} \frac{\ln\left(1+\frac{t}{2}\right)}{t} = 0$$

TCO $\Rightarrow$

$$\lim_{\|(x,y)\|\to+\infty} f_{|C}(x,y) = 0$$

Cerco una restrizione di f che $\underline{\text{non}}$ tende a 0 all'infinito



Provo con

$$D := \left\{ (x,y) \in \mathrm{dom}(f) \;\middle|\; y = \underbrace{-\frac{1}{x} + \frac{1}{x^2}}_{-\frac{1}{x} < \quad < 0}, \quad x > 1 \right\}$$

$$f_{|D}(x,y) = f\left(x, -\frac{1}{x} + \frac{1}{x^2}\right) = \frac{\ln\left(1 + x\left(-\frac{1}{x}+\frac{1}{x^2}\right)\right)}{x^2 + \left(-\frac{1}{x}+\frac{1}{x^2}\right)^2}$$

$$= \frac{\overbrace{\ln\left(\overbrace{\tfrac{1}{x}}^{\to 0}\right)}^{-\infty}}{\underbrace{\underbrace{x^2}_{\to+\infty} + \underbrace{\left(-\frac{1}{x}+\frac{1}{x^2}\right)^2}_{\to 0}}_{\to+\infty}} \longrightarrow 0 \qquad ?? \quad \ln\left(\frac{1}{x}\right) = -\ln(x)$$

$x \to +\infty$

Morale: devo "avvicinarmi di più alla frontiera"

Provo con $E = \left\{ (x,y) \in \mathrm{dom}(f) \;\middle|\; y = -\frac{1}{x} + \frac{e^{-x^2}}{x} \right\}$

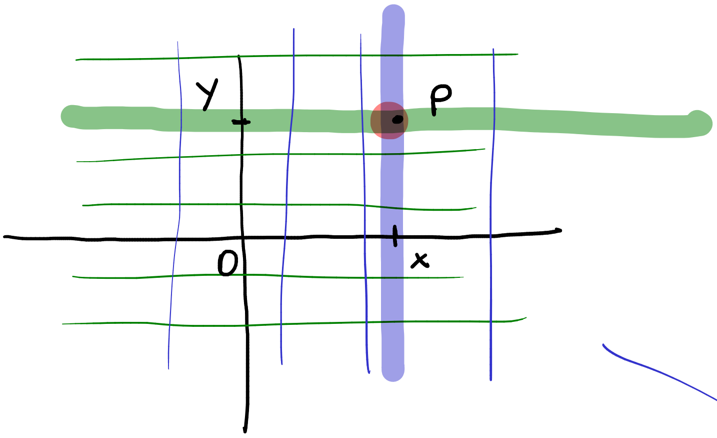
$-\frac{1}{x} < \quad < 0$, $\quad e^{-x^2} < 1$

$$F|_E(x,y) = \frac{\ln\left(1 + x\left(-\frac{1}{x} + \frac{e^{-x^2}}{x}\right)\right)}{x^2 + \left(-\frac{1}{x} + \frac{e^{-x^2}}{x}\right)^2}$$

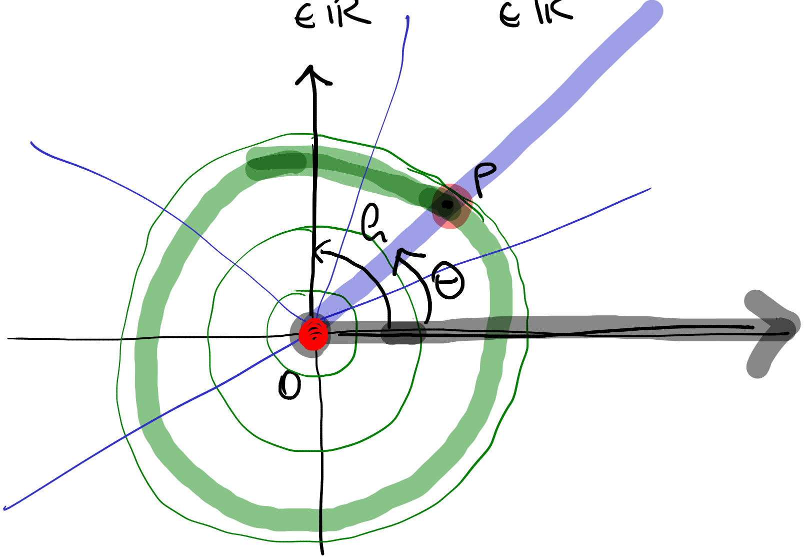
$$= \frac{\ln\left(e^{-x^2}\right)}{x^2 + (\dots)^2} = \frac{-x^2}{x^2 + \underbrace{(\dots)^2}_{\to 0}} \underset{x \to +\infty}{\sim} \frac{-x^2}{x^2} \to \underbrace{-1}_{\neq 0}$$

$$\Rightarrow \nexists \lim_{\|(x,y)\| \to +\infty} f(x,y) \qquad \square$$

Richiamo sulle coordinate polari nel piano



$P \equiv (x, y)$ con $x \in \mathbb{R}$, $y \in \mathbb{R}$



$P \equiv (\rho, \theta)$

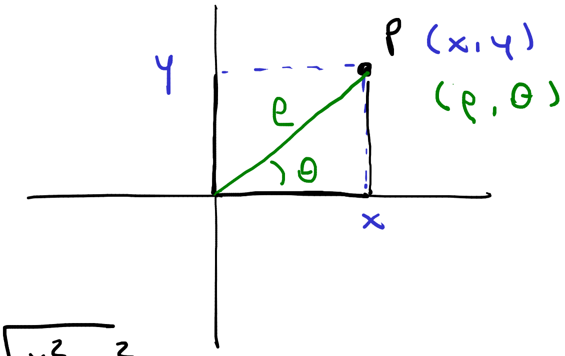
$\rho \in [0, +\infty)$, $\theta \in [0, 2\pi)$ $\quad (\in [-\pi, \pi))$

Oss: corrispondenza non biunivoca tra

$(\rho, \theta) \in [0, +\infty) \times [0, 2\pi)$ e il piano

(Problema: l'origine)

Qual è la relazione tra coord. cartesiane e coord. polari ?



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\rho^2 = x^2+y^2 \Rightarrow \rho = \sqrt{x^2+y^2}$$

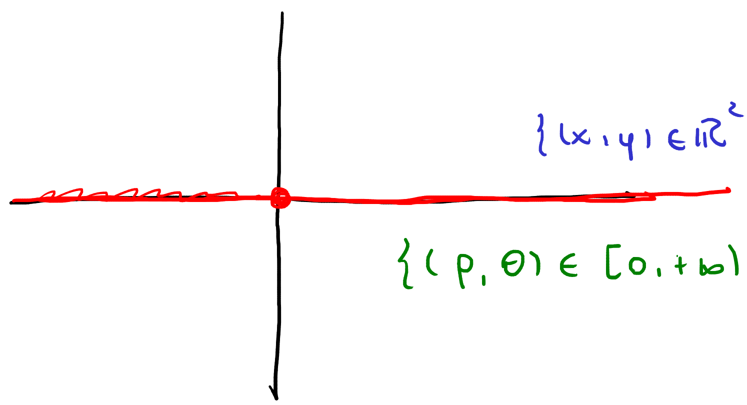
$\cancel{x \neq 0 \quad \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta}$

$\cancel{\theta = \underbrace{\arctan\left(\frac{y}{x}\right)}_{\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}}$!!!

θ è l'unica soluzione in $[0,2\pi)$ (oppure $[-\pi,\pi)$) del sistema

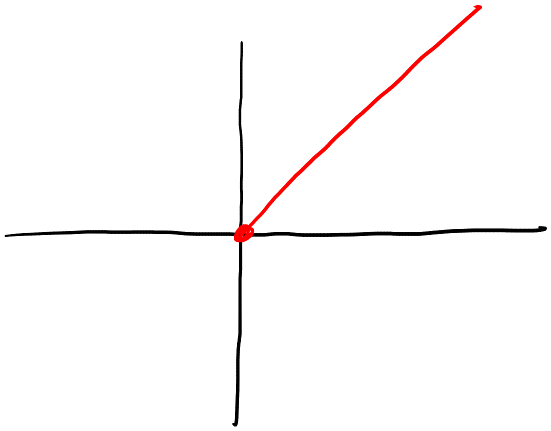
$$\begin{cases} \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}} \in [-1,1] \\ \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}} \in [-1,1] \end{cases}$$

"Traduco" da cartesiano a polare alcuni sottoinsiemi di $\mathbb{R}^2$



$\{(x,y)\in\mathbb{R}^2 \mid y=0\}$

$\{(\rho,\theta)\in[0,+\infty)\times[0,2\pi) \mid \theta=0 \text{ oppure } \theta=\pi\}$

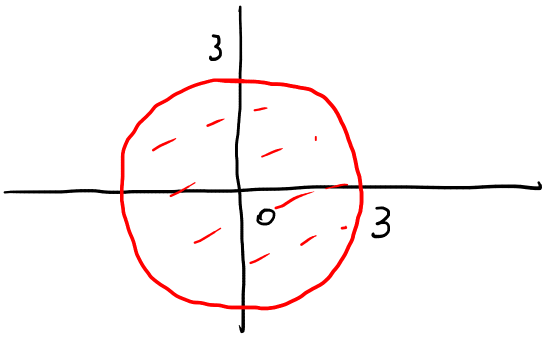


$\{(x,y)\in\mathbb{R}^2 \mid y=x,\ \ x\geq 0\}$

$\{(\rho,\theta)\in[0,+\infty)\times[0,2\pi) \mid \theta=\frac{\pi}{4}\}$

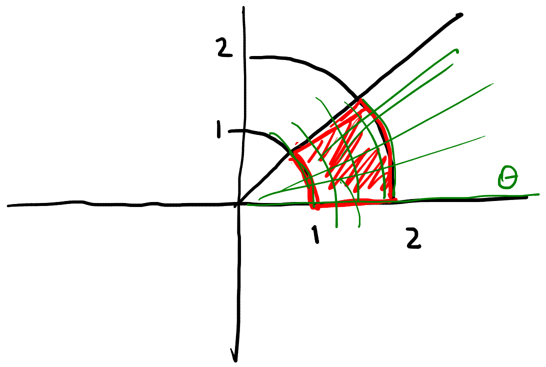
$(x\neq 0)$ $\cos\theta = \frac{x}{\sqrt{x^2+x^2}} = \frac{x}{\sqrt{2}\,|x|} = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}}$

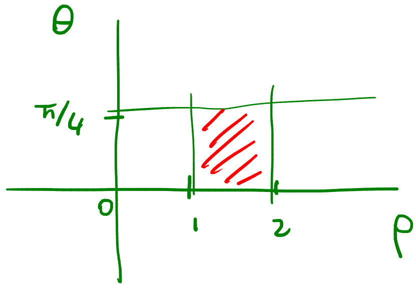


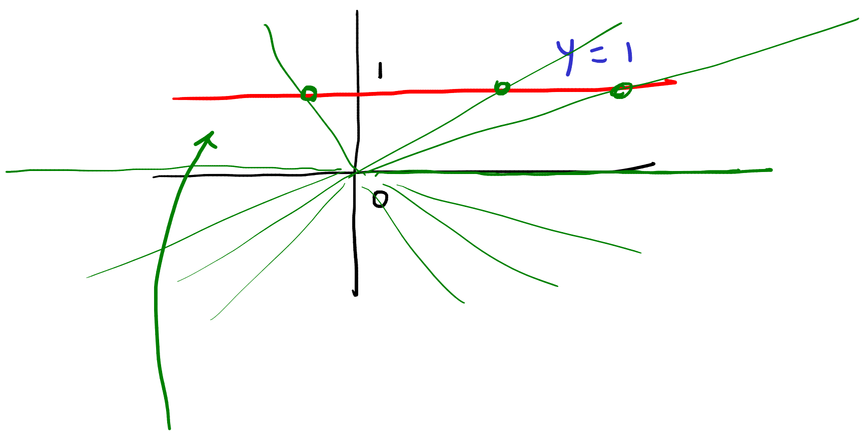
$x^2+y^2 \underset{\leq}{=} 9$

$\rho \underset{\leq}{=} 3$



$\{1\leq\rho\leq 2\ ,\ \ 0\leq\theta\leq\frac{\pi}{4}\}$





$0 < \theta < \pi$

$\rho\sin\theta = 1$

$\rho = \frac{1}{\sin\theta}$ $\neq 0$

$\{(\rho,\theta) \mid 0<\theta<\pi,\ \ \rho=\frac{1}{\sin\theta}\}$

Es:

- $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\qquad \mathbb{R}^2 \setminus \{(0,0)\}$

$$? \lim_{(x,y)\to(0,0)} f(x,y)$$

se esistesse, sarebbe 0

Coord. polari di centro $(0,0)$:

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta \qquad (\rho>0)$$

$$g(\rho,\theta) := f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho\cos\theta \ \rho\sin\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta}$$

$$= \frac{\rho^2\cos\theta\sin\theta}{\rho^2} = \cos\theta\sin\theta$$

$$\left|\tfrac{1}{2}\sin 2\theta\right|$$

$$\sup_{\theta\in[0,2\pi]} |g(\rho,\theta) - 0| = \sup_{\theta\in[0,2\pi]} |\cos\theta\sin\theta| = \frac{1}{2}$$

non dipende da ρ

$$\lim_{\rho\to 0} \sup_{\theta\in[0,2\pi]} |g(\rho,\theta) - 0| = \frac{1}{2} \neq 0$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

- $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$ $\qquad (x,y)\to(0,0)$

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta \qquad (\rho>0)$$

$$g(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta)$$

$$= \frac{\rho^2\cos^2\theta \ \rho\sin\theta}{\rho^2} = \rho\cos^2\theta\sin\theta$$

$$\forall (\rho,\theta): \quad |g(\rho,\theta)| = |\rho\cos^2\theta\sin\theta|$$

$$= \rho \underbrace{\cos^2\theta}_{\leq 1} \underbrace{|\sin\theta|}_{\leq 1} \leq \rho$$

$$\Rightarrow 0 \leq \sup_{\theta\in[0,2\pi]} |g(\rho,\theta) - 0| \leq \underset{0}{\underset{\downarrow}{\rho}} \quad \rho \to 0$$

$$\overset{TCO}{\Rightarrow} \lim_{\rho\to 0} \sup_{\theta\in[0,2\pi]} |g(\rho,\theta) - 0| = 0$$

$$\Rightarrow \exists \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

ES.

- Studio la continuità di:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y)\in\mathbb{R}^2\setminus\{(0,0)\} \\ 0 & (x,y)=(0,0) \end{cases}$$

$f|_{\mathbb{R}^2\setminus\{(0,0)\}}$ è una funzione razionale

$\Rightarrow$ è continua

Verifico se f è continua in $(0,0)$, cioè se

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \quad (=0)$$

Utilizzo coord. polari:

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta$$

Per $\rho>0$: $\quad g(\rho,\theta) := f(\rho\cos\theta, \rho\sin\theta)$

$$= \frac{\rho\cos\theta\, \rho\sin\theta\, (\rho^2\cos^2\theta - \rho^2\sin^2\theta)}{\rho^2}$$

$$= \rho^2 \cos\theta\sin\theta\,(\cos^2\theta - \sin^2\theta)$$

Oss: $|\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)|$

$$\le |\cos^2\theta - \sin^2\theta| \le |\cos^2\theta| + |\sin^2\theta| \le 2$$

$$\Rightarrow 0 \le \sup_{\theta\in[0,2\pi]} |g(\rho,\theta)| \le \underbrace{2\rho^2}_{\to 0} \quad \rho \to 0$$

$$\overset{TCo}{\Rightarrow} \lim_{\rho\to 0} \sup_{\theta\in[0,2\pi]} |g(\rho,\theta)| = 0$$

$$\Rightarrow \exists \lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad (= f(0,0))$$

$\Rightarrow$ f è continua anche in $(0,0)$. $\square$

- Stabilire se

$$f(x,y) = \begin{cases} \dfrac{1-\sqrt{1-x^2-y^2}}{x^2+y^2} & (x,y) \neq (0,0) \\ \dfrac{1}{2} & (x,y) = (0,0) \end{cases}$$

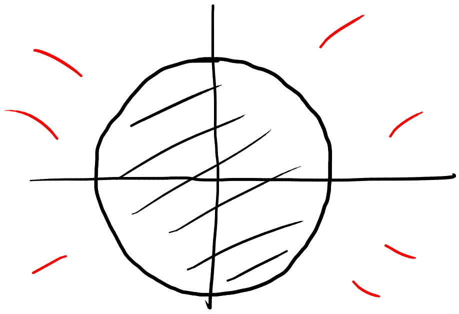
ammette estremi globali nel proprio dominio.

$$\mathrm{dom}(f) = \{(x,y)\in\mathbb{R}^2 \setminus \{(0,0)\} \mid 1-x^2-y^2 \ge 0\} \cup \{(0,0)\}$$

$$= \{ \quad '' \quad \mid x^2+y^2 \le 1\} \cup \{(0,0)\}$$

$$= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 1\}$$

OSS: $\mathrm{dom}(f)$ è chiuso e limitato, quindi compatto



Per il teor. di Weierstrass, stabilire se f ha estremi globali equivale a stabilire se f è continua.

$f|_{\mathrm{dom}(f)\setminus\{(0,0)\}}$ è continua perché composta di funzioni continue.

Verifico se f è continua in $(0,0)$.

Calcolo

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1-\sqrt{1-(x^2+y^2)}}{x^2+y^2} \qquad \frac{0}{0} \; !!$$

$$\underset{t = x^2+y^2}{=} \lim_{t\to 0} \frac{1-\sqrt{1-t}}{t} = \lim_{t\to 0} \frac{\frac{1}{2}t + o(t)}{t} = \frac{1}{2} = f(0,0)$$

$g(t) := 1-\sqrt{1-t} \qquad t \in (-1,1)$

$g'(t) = -\dfrac{1}{2\sqrt{1-t}}(-1)$

$g(0) = 0 \qquad g'(0) = \frac{1}{2}$

$\Rightarrow \; g(t) = g(0) + g'(0)\,t + o(t) = 0 + \frac{1}{2}t + o(t)$

$\Rightarrow$ f è continua in $(0,0)$, dunque in tutto il dominio. $\square$

ES.

$$r(t) = x + t(y-x) \qquad t \in \mathbb{R}$$

$$\forall i: \quad r_i(t) = \underbrace{x_i}_{\text{cost.}} + \underline{t}\,(\underbrace{y_i - x_i}_{\text{cost.}}) \qquad r_i : \mathbb{R} \to \mathbb{R} \text{ continua}$$

banalità ..

$\Rightarrow$ r continua