25 marzo 2025

Verifico che la funzione $d : X \times X \to \mathbb{R}_+$ t.c.

$$d(x,y) := \|x - y\|$$

è una metrica in X.

(D1) $x, y \in X$

$$d(x,y) = 0 \overset{\text{def}}{\iff} \|x-y\| = 0 \overset{(N1)}{\iff} x - y = 0$$

$$\iff x = y \quad \checkmark$$

(D2) $x, y \in X$

$$d(y,x) \overset{\text{def}}{=} \|y - x\| = \|-(x-y)\|$$

$$= \|(-1)(x-y)\| \overset{(N2)}{=} |-1| \, \|x-y\|$$

$$= \|x-y\| \overset{\text{def}}{=} d(x,y) \quad \checkmark$$

(D3) $x, y, z \in X$

$$d(x,y) \overset{\text{def}}{=} \|x - y\| = \|\underline{x - z} + \underline{z - y}\|$$

$$\overset{(N3)}{\leq} \|x - z\| + \|z - y\| \overset{\text{def}}{=} d(x,z) + d(z,y) \checkmark$$

$\square$

Esempi

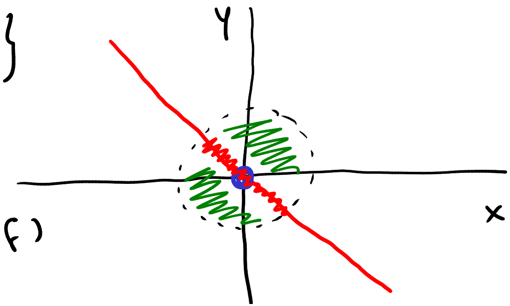
- Verifico che $\displaystyle\lim_{(x,y)\to(0,0)} \overbrace{(x^2+y^2)\sin\left(\frac{1}{x+y}\right)}^{=: f(x,y)} = 0$

$$\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 \mid x + y \neq 0\}$$

↑
dominio di f

$(0,0) \notin \text{dom}(f)$

$\Rightarrow \text{dom}(f) \setminus \{(0,0)\} = \text{dom}(f)$



$(0,0) \in Dr(\mathrm{dom}(f))$ ✓

Fisso $((x_k, y_k)) \subset \mathrm{dom}(f)$ tale che

$$(x_k, y_k) \to (0,0)$$

cioè : $x_k \to 0, \quad y_k \to 0$.

Valuto la successione trasformata:

$$f(x_k, y_k) = (\underbrace{x_k^2}_{\downarrow\, 0} + \underbrace{y_k^2}_{\downarrow\, 0}) \sin\left(\frac{1}{\underbrace{\underbrace{x_k}_{\to 0} + \underbrace{y_k}_{\to 0}}_{\to 0}} \right)$$

!!!

Ricordo dall'analisi I che la successione prodotto di una succ. infinitesima per una succ. limitata è una succ. infinitesima.

Quindi: $\left.\begin{array}{l} x_k^2 + y_k^2 \to 0 \\ \underbrace{\left|\sin\left(\frac{1}{x_k + y_k}\right)\right|}_{\text{limitata}} \leq 1 \end{array}\right\} \Rightarrow f(x_k, y_k) \to 0$

$\square$

- $$\lim_{(x,y)\to(0,0)} \frac{3x^3 + 2x^2 + 2y^2}{x^2 + y^2} = 2$$

$f(x,y) := \dfrac{3x^3 + 2x^2 + 2y^2}{x^2+y^2}$ $\qquad \mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

$(0,0) \in Dr(\mathrm{dom}(f))$

Fisso $((x_k, y_k)) \subset \mathbb{R}^2 \setminus \{(0,0)\}$ t.c. $(x_k, y_k) \to (0,0)$ (cioè : $x_k \to 0$, $y_k \to 0$) . Valuto la succ. trasformata:

$$f(x_k, y_k) = \frac{\overbrace{3x_k^3}^{\to 0} + \overbrace{2x_k^2}^{\to 0} + \overbrace{2y_k^2}^{\to 0}}{\underbrace{x_k^2}_{\to 0} + \underbrace{y_k^2}_{\to 0}}$$

forma di indecisione $\frac{0}{0}$

$$= \frac{3x_k^3}{x_k^2 + y_k^2} + \frac{2(\cancel{x_k^2 + y_k^2})}{\cancel{x_k^2 + y_k^2}} = 2$$

Verifico che $\frac{3x_k^3}{x_k^2 + y_k^2} \to 0$

$$x_k^2 \le \overbrace{x_k^2 + \underbrace{y_k^2}_{\geq 0}}^{\geq 0}$$

Osservo che $\forall k$:

$$\underbrace{0}_{\to 0} \le \left| \frac{3x_k^3}{x_k^2 + y_k^2} \right| = \frac{3\,|x_k|^3}{x_k^2 + y_k^2} = \underbrace{3|x_k|}_{\geq 0} \underbrace{\frac{x_k^2}{x_k^2 + y_k^2}}_{\le 1}$$

$$\le \underbrace{3|x_k|}_{\to 0}$$

TCO $\Rightarrow$ $\left| \frac{3x_k^3}{x_k^2 + y_k^2} \right| \to 0 \iff \frac{3x_k^3}{x_k^2 + y_k^2} \to 0$

Perciò: $f(x_k, y_k) = \frac{3x_k^3}{x_k^2 + y_k^2} + 2 \longrightarrow 0 + 2 = 2$

- $$\lim_{(x,y) \to (1,1)} \underbrace{\frac{(y-1)^4}{x^2 + y^2 + 2(1 - x - y)}}_{=: f(x,y)}$$

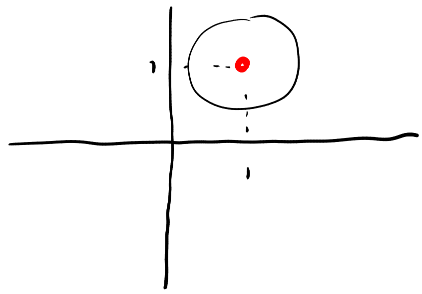
$$x^2 + y^2 + 2(1 - x - y) = x^2 + y^2 + 1 + 1 - 2x - 2y$$

$$= (x-1)^2 + (y-1)^2$$

$$\Rightarrow \quad f(x,y) = \frac{(y-1)^4}{(x-1)^2 + (y-1)^2} \qquad = \|(x,y) - (1,1)\|^2_{\mathbb{R}^2} = d_{\mathbb{R}^2}((x,y),(1,1))$$

$$\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(1,1)\}$$

$$(1,1) \in \mathrm{Dr}\,(\mathrm{dom}(f))$$



Fisso $(x_k, y_k) \subset \mathbb{R}^2 \setminus \{(1,1)\}$ t.c.

$$(x_k, y_k) \to (1,1) \qquad (\text{cioè } x_k \to 1,\ y_k \to 1)$$

e valuto

$$f(x_k, y_k) = \frac{\overbrace{(y_k-1)^4}^{\to 0}}{\underbrace{(x_k-1)^2}_{\to 0} + \underbrace{(y_k-1)^2}_{\to 0}} \qquad \frac{0}{0} \ !!$$

Risulta $\forall k$:

$$\underset{\downarrow\, 0}{0} \le f(x_k, y_k) = \underbrace{(y_k-1)^2}_{\geq 0} \underbrace{\frac{(y_k-1)^2}{(x_k-1)^2 + (y_k-1)^2}}_{\le 1} \le \underset{\downarrow\, 0}{(y_k-1)^2} \quad y_k \to 1$$

$$\overset{\text{TCO}}{\Rightarrow} \quad f(x_k, y_k) \to 0. \qquad \square$$

- $$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^2} \overset{?}{=} 0$$

$$f(x,y) = \frac{x^3 y}{x^4 + y^2} \qquad \mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$$

$$(0,0) \in \mathrm{Dr}\,(\mathrm{dom}(f))$$

Fisso $(x_k, y_k) \subset \mathbb{R}^2 \setminus \{(0,0)\}$ t.c. $(x_k, y_k) \to (0,0)$

e valuto $f(x_k, y_k) = \dfrac{x_k^3 \, y_k}{x_k^4 + y_k^2}$

Oss : $\forall a, b \in \mathbb{R}$:

$$0 \le (|a| - |b|)^2 = a^2 + b^2 - 2|a||b|$$

$$\Rightarrow \quad 2|a||b| \le a^2 + b^2$$

$$\Rightarrow \quad |a||b| \le \frac{a^2+b^2}{2} \quad \Rightarrow \quad \frac{|a||b|}{a^2+b^2} \le \frac{1}{2}$$

Riprendo

$$f(x_k, y_k) = \frac{x_k^3 \, y_k}{x_k^4 + y_k^2} = x_k \frac{\overbrace{x_k^2}^{a} \, \overbrace{y_k}^{b}}{\underbrace{x_k^4}_{a^2} + \underbrace{y_k^2}_{b^2}}$$

$$\Rightarrow \quad \underset{\downarrow \atop 0}{0} \le |f(x_k, y_k)| = |x_k| \underbrace{\frac{|x_k^2| \, |y_k|}{x_k^4 + y_k^2}}_{\le \frac{1}{2}} \le \underbrace{\underset{\downarrow \atop 0}{|x_k|} \frac{1}{2}}_{\downarrow \atop 0}$$

TCO $\Rightarrow$ $f(x_k, y_k) \to 0$ $\square$

• $\displaystyle \lim_{(x,y,z) \to (0,0,0)} \underbrace{\frac{(x+y)\, z^4}{x^4 + y^2 + z^4}}_{=: f(x,y,z)}$

$$\mathrm{dom}(f) = \mathbb{R}^3 \setminus \{(0,0,0)\}$$

Fisso $((x_k, y_k, z_k)) \subset \mathbb{R}^3 \setminus \{(0,0,0)\}$ t.c.

$$(x_k, y_k, z_k) \to (0,0,0) \quad :$$

$$0 \le |f(x_k, y_k, z_k)| = |x_k + y_k| \underbrace{\left(\frac{z_k^4}{x_k^4 + y_k^2 + z_k^4}\right)}_{\le 1}$$

$$\le \underbrace{|\overbrace{\underset{\to 0}{x_k} + \underset{\to 0}{y_k}}^{\to 0}|}_{\to 0}$$

$$\overset{\text{TCo}}{\Longrightarrow} \quad f(x_k, y_k, z_k) \to 0 . \qquad \square$$

Cercare $(x_k, y_k) \subset \mathbb{R}^2$ t.c.

- $x_k^2 + y_k^2 \to +\infty \qquad \left(\|(x_k, y_k)\|^{\cancel{2}} \to +\infty \right)$
- $|x_k| \not\to +\infty, \quad |y_k| \not\to +\infty$