13 marzo 2025

Verifico che la funzione $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ definita ponendo

$$d(x,y) := |x-y|$$

è una metrica in $\mathbb{R}$.

D1 : $x, y \in \mathbb{R}$

$$d(x,y) = 0 \iff |x-y| = 0 \iff x-y=0$$
$$\iff x = y \quad \checkmark$$

D2 : $x, y \in \mathbb{R}$

$$d(y,x) \overset{\text{def}}{=} |y-x| = |-(x-y)| = |x-y| \overset{\text{def}}{=} d(x,y) \quad \checkmark$$

D3 : $x, y, z \in \mathbb{R}$

$$d(x,y) \overset{\text{def}}{=} |x-y| = |\underbrace{x-z}_{a} + \underbrace{z-y}_{b}|$$

$$\leq |x-z| + |z-y| \underset{\text{def}}{=} d(x,z) + d(z,y). \quad \checkmark$$

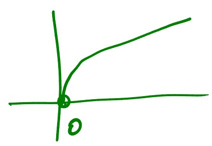
$\uparrow$ dis. triang. di $|\cdot|$

$\square$

Verifico che la funzione $d_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ tale che

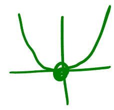
$$d_{\mathbb{R}^n}(x,y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

soddisfa (D1) e (D2).



D1 : $d_{\mathbb{R}^n}(x,y) = 0 \overset{\text{def}}{\iff} \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$

$$(x_i - y_i)^2 \geq 0$$



$$\Leftrightarrow \sum_{i=1}^{n} (x_i - y_i)^2 = 0 \Leftrightarrow \forall i \in \{1, \dots, n\} : (x_i - y_i)^2 = 0$$

$$\Leftrightarrow \forall i \in \{1, \dots, n\} : x_i - y_i = 0$$

$$\Leftrightarrow \forall i \in \{1, \dots, n\} : x_i = y_i \Leftrightarrow x = y \quad \checkmark$$

$$D2: \quad d_{\mathbb{R}^n}(x,y) \overset{def}{=} \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

$$\overset{def}{=} d_{\mathbb{R}^n}(y,x) \quad \checkmark \qquad \square$$

Oss: per $n = 1$:

$$d_{\mathbb{R}^1}(x,y) = \sqrt{(x-y)^2} = |x-y|$$

Verifico che d_1 è una metrica

(D1) $x, y \in \mathbb{R}^n$

$$|x_i - y_i| \geq 0$$

$$d_1(x,y) = 0 \overset{def}{\Leftrightarrow} \sum_{i=1}^{n} |x_i - y_i| = 0 \Leftrightarrow$$

$$\forall i : |x_i - y_i| = 0 \Leftrightarrow \forall i : x_i - y_i = 0$$

$$\Leftrightarrow \forall i : x_i = y_i \Leftrightarrow x = y \quad \checkmark$$

(D2) $x, y \in \mathbb{R}^n$:

$$d_1(y,x) \overset{def}{=} \sum_{i=1}^{n} |y_i - x_i| = \sum_{i=1}^{n} |x_i - y_i| = d_1(x,y) \quad \checkmark$$

(D3) $x, y, z \in \mathbb{R}^n$

$$d_1(x,y) \overset{def}{=} \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$$

$\forall i : |x_i - z_i + z_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$

+ compatibilità di addizione e relazione d'ordine

$$\leq \sum_{i=1}^{n} \left(|x_i - z_i| + |z_i - y_i| \right)$$

$$= \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i| \overset{\text{def}}{=} d_1(x,z) + d_1(z,y) \quad \checkmark$$

Verifico che d_{max} è una metrica in $\mathbb{R}^n$.

D1 : $d_{max}(x,y) = 0 \overset{\text{def}}{\iff} \max\limits_{1 \leq i \leq n} \underbrace{|x_i - y_i|}_{\geq 0} = 0$

$\iff \forall i : |x_i - y_i| = 0 \iff \ldots \iff x = y \quad \checkmark$

D2 : $d_{max}(y,x) = \max\limits_{1 \leq i \leq n} |y_i - x_i| = \max\limits_{1 \leq i \leq n} |x_i - y_i|$

$= d_{max}(x,y)$

D3 : $x, y, z \in \mathbb{R}^n$

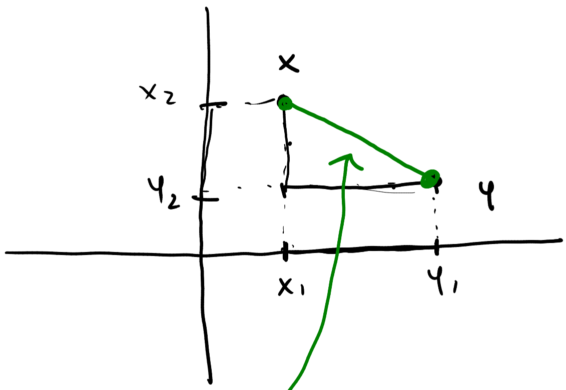
$\forall i : |x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$

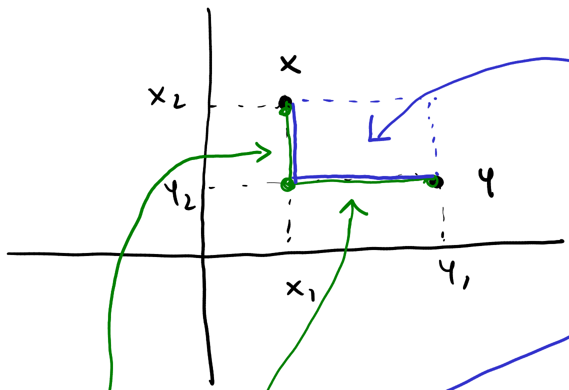
$\leq \underbrace{d_{max}(x,z) + d_{max}(z,y)}$

è un maggiorante di $|x_i - y_i|$ al variare di $i \in \{1, \ldots, n\}$

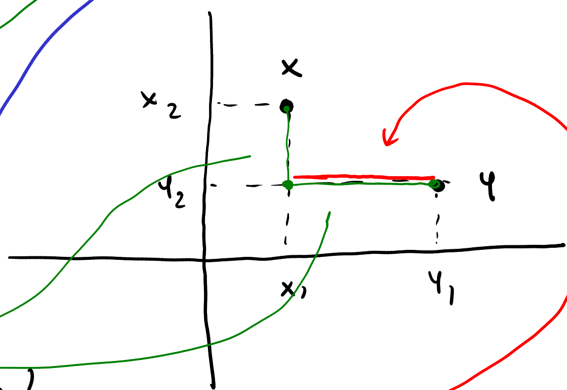
$\Rightarrow \underbrace{\max\limits_{1 \leq i \leq n} |x_i - y_i|}_{=: d_{max}(x,y)} \leq d_{max}(x,z) + d_{max}(z,y) \quad \checkmark$

"Visualizzo" le tre metriche per $n = 2$





$$d_{\mathbb{R}^2}(x,y) \overset{def}{=} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$



$$d_1(x,y) \overset{def}{=} |x_1-y_1| + |x_2-y_2|$$

$$d_{max}(x,y) \overset{def}{=} \max\{|x_1-y_1|, |x_2-y_2|\}$$

Verifico che la funzione

$$d_{DIS}(x,y) := \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

soddisfa D3:

$x, y, z \in X$

$$d(x,y) \overset{?}{\leq} d(x,z) + d(z,y) \quad (*)$$

Se $x = y$: $d(x,y) = 0 \Rightarrow (*)$ è vera perché $d(x,z) \geq 0$, $d(z,y) \geq 0$

Se $x \neq y$: $d(x,y) = 1$, quindi

$(*) \iff 1 \leq d(x,z) + d(z,y) \quad (**)$

Dato che $x \neq y$, almeno uno tra $d(x,z)$ e $d(z,y)$ deve essere uguale a 1, quindi $(**)$ è vera. $\square$

Esempi di intorni sferici:

- $(X, d_{DIS}) \quad x_0 \in X, \quad r > 0$

$$B_r(x_0) \overset{def}{=} \{x \in X \mid d_{DIS}(x, x_0) < r\}$$

$$= \begin{cases} \{x \in X \mid d_{DIS}(x,x_0) = 0\} = \{x_0\} & 0 < r \leq 1 \\ X & r > 1 \end{cases}$$

- $\mathbb{R}$ con metrica del valore assoluto

$x_0 \in \mathbb{R}, \quad r > 0$

$$B_r(x_0) \overset{def}{=} \{x \in \mathbb{R} \mid |x - x_0| < r\}$$

$$= \{x \in \mathbb{R} \mid x_0 - r < x < x_0 + r\}$$

$$= (x_0 - r, x_0 + r)$$ intervallo aperto di centro x_0 e semiampiezza r

- $(\mathbb{R}^2, d_{\mathbb{R}^2})$

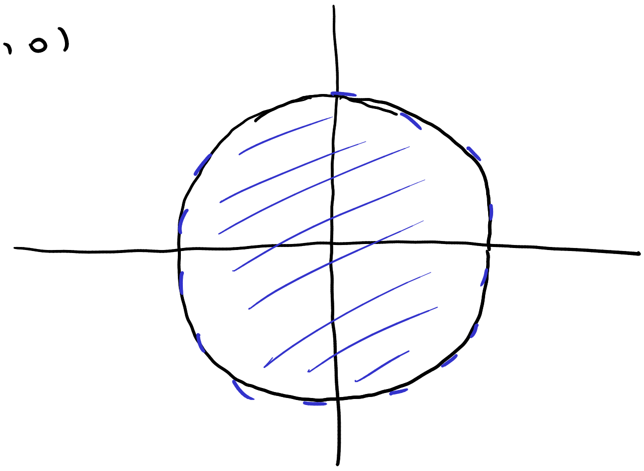
$(x_0, y_0) \in \mathbb{R}^2, \quad r > 0$

$$B_r(x_0, y_0) \overset{def}{=} \{(x,y) \in \mathbb{R}^2 \mid d_{\mathbb{R}^2}((x,y),(x_0,y_0)) < r\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\}$$

$= \{ (x,y) \in \mathbb{R}^2 \mid (x-x_0)^2 + (y-y_0)^2 < r^2 \}$

Per es: $(x_0, y_0) = (0,0)$



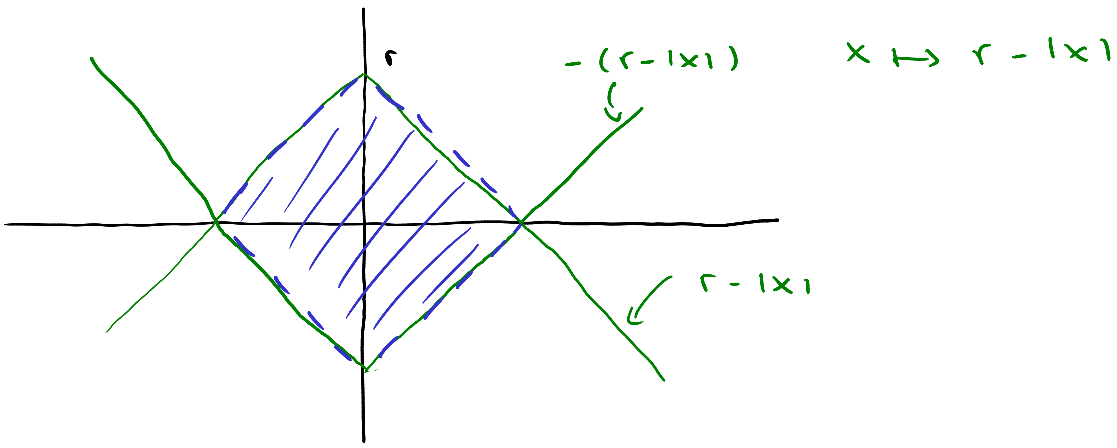
In $(\mathbb{R}^2, d_1)$:

$$B_r(x_0,y_0) \overset{def}{=} \{ (x,y) \in \mathbb{R}^2 \mid |x-x_0| + |y-y_0| < r \}$$

Per $(x_0,y_0) = (0,0)$:

$$|x| + |y| < r \iff |y| < r - |x|$$

$$\iff -(r-|x|) < y < r - |x|$$



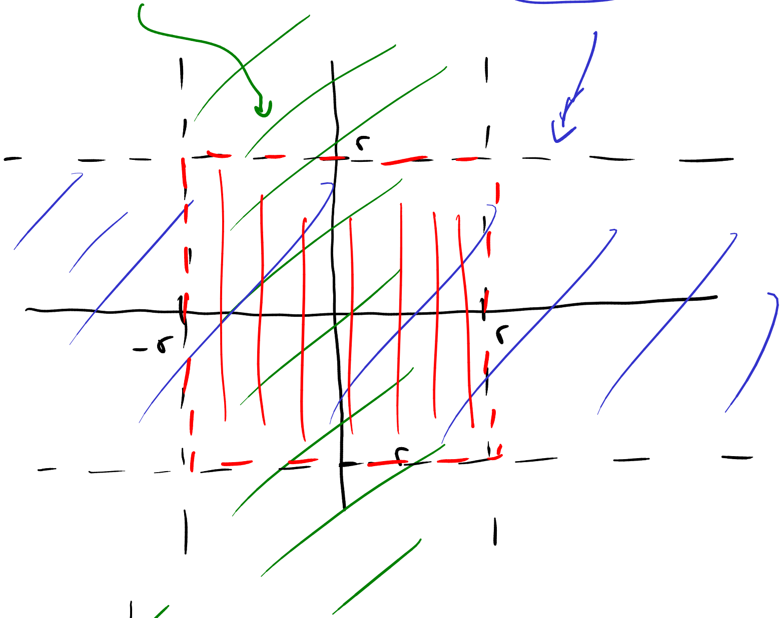
In $(\mathbb{R}^2, d_{max})$:

$$B_r(x_0,y_0) = \{ (x,y) \in \mathbb{R}^2 \mid \max\{|x-x_0|, |y-y_0|\} < r \}$$

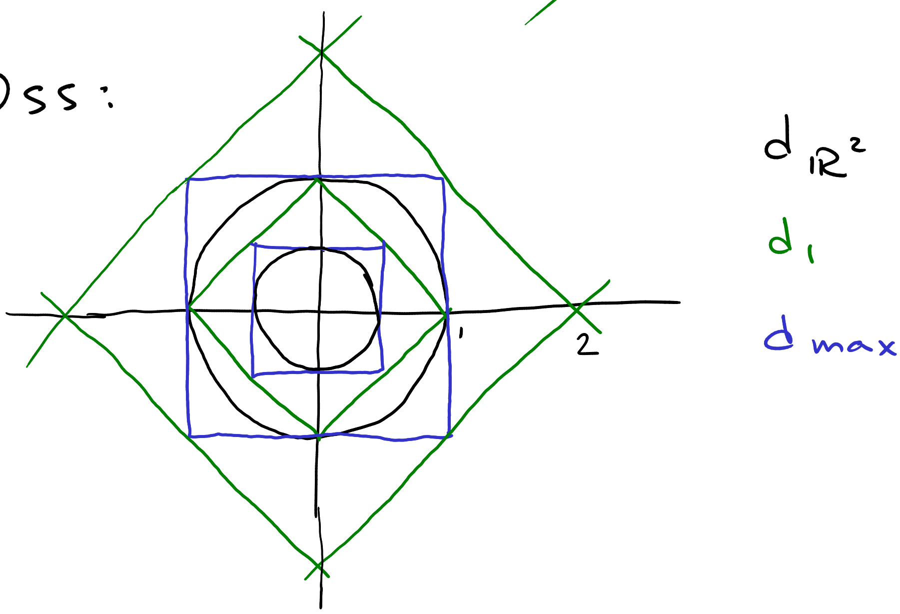
Per $(x_0,y_0) = (0,0)$:

$\max\{|x|, |y|\} < r$

$\iff$ $|x| < r$ e $|y| < r$



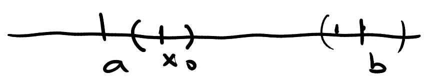
Oss:



Es: $E_1 = (a,b)$, $E_2 = [a,b]$, $E_3 = (a,b]$, $E_4 = [a,b)$

Punti interni: $\forall x$ t.c. $a < x < b$

Punti esterni:



$\forall x$ t.c. $x < a$ oppure $x > b$

Punti di frontiera: a, b

Punti di accumulazione: $\forall x \in [a,b]$