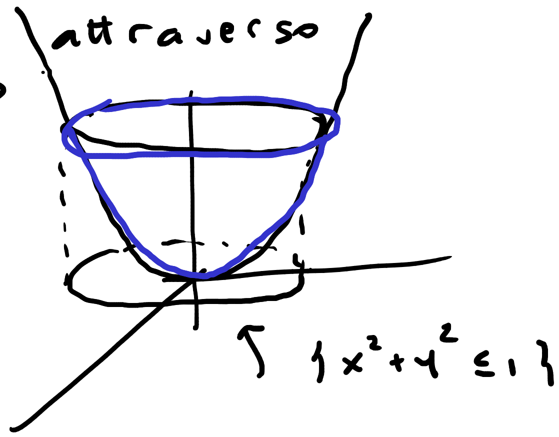
6 giugno 2024

Esempio (flusso)

$F(x,y,z) = (y, x, z)$

Flusso diretto verso il basso attraverso

$f(x,y) = x^2+y^2$



Parametrizzo Σ:

$\sigma(u,v) = (u, v, u^2+v^2)$

$(u,v) \in \bar{B}_1(0,0)$

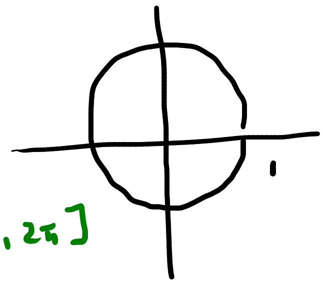
So che: $N_\sigma(u,v) = (-2u, -2v, \underbrace{1}_{>0})$ punta verso l'alto

$$- \Phi_\Sigma(F) = \iint_{\bar{B}_1(0,0)} F(\sigma(u,v)) \cdot N_\sigma(u,v)\, du\, dv$$

$$= \iint_{\bar{B}_1(0,0)} F(u, v, u^2+v^2) \cdot (-2u, -2v, 1)\, du\, dv$$

$$= \iint_{\bar{B}_1(0,0)} (v, u, u^2+v^2) \cdot (-2u, -2v, 1)\, du\, dv$$

$$= \iint_{\bar{B}_1(0,0)} (-2uv - 2uv + u^2+v^2)\, du\, dv$$



$u = \rho\cos\theta$

$v = \rho\sin\theta$

$(\rho,\theta) \in [0,1] \times [0, 2\pi]$

$$= \iint_{[0,1]\times[0,2\pi]} (-4\rho\cos\theta \cdot \rho\sin\theta + \rho^2\cos^2\theta + \rho^2\sin^2\theta)\, \underbrace{\rho}\, d\rho\, d\theta$$

$$= \iint_{[0,1]\times[0,2\pi]} \rho^3 (1 - 4\cos\theta \sin\theta)\, d\rho\, d\theta$$

$$= \int_0^1 \rho^3\, d\rho \cdot \int_0^{2\pi} (1 - \underbrace{2\sin(2\theta)}_{\to \int \dots = 0})\, d\theta$$

$$= \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

$$\Rightarrow \quad \Phi_\Sigma(F) = -\frac{\pi}{2}. \qquad \square$$

Esempi (sul rotore)

- $F(x,y,z) = (xy,\ x^2,\ yz) \qquad (x,y,z) \in \mathbb{R}^3$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & yz \end{vmatrix} \qquad \operatorname{rot} F(x,y,z) = (z - 0,\ -(0-0),\ 2x - x)$$

$$= (z,\ 0,\ x)$$

- $F(x,y,z) = (x,y,z)$

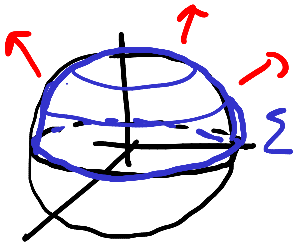
$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \qquad \operatorname{rot}(F)(x,y,z) = (0,0,0)$$

- $F(x,y,z) = (y,\ -x,\ 0)$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} \qquad \operatorname{rot} F(x,y,z) = (0,\ 0,\ -1-1)$$

$$= (0,0,-2)$$

Esempio (teor. di Stokes)



$$F(x,y,z) = (xy,\ x^2,\ yz)$$

Già calcolato: $\operatorname{rot} F(x,y,z) = (z, 0, x)$

Parametrizzo Σ:

$$\sigma(\varphi,\theta) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

$$(\varphi,\theta) \in \left[0, \tfrac{\pi}{2}\right] \times [0, 2\pi] =: K$$

Già visto: $N_\sigma(\varphi,\theta) = (\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \underbrace{\sin\varphi\cos\varphi}_{>0})$

punta verso l'alto

$\varphi \in \left[0, \frac{\pi}{2}\right]$

1° membro:

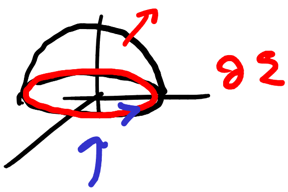
$$\iint_\Sigma \operatorname{rot} F \cdot n \, dS =$$

$$\iint_K (\overset{z}{\cos\varphi},\ \overset{0}{0},\ \overset{x}{\sin\varphi\cos\theta}) \cdot (\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \iint_K (\cos\varphi\sin^2\varphi\cos\theta + 0 + \sin^2\varphi\cos\varphi\cos\theta)\, d\varphi\, d\theta$$

$$= \iint_K 2\cos\varphi\sin^2\varphi\cos\theta \, d\varphi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\cos\varphi\sin^2\varphi\, d\varphi \cdot \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{=0} = 0$$



$r(t) = (\cos t, \sin t, 0) \quad t \in [0, 2\pi]$

2° membro: $\omega = xy\,dx + x^2\,dy + yz\,dz$

$$\int_{\partial\Sigma^+} F(P)\cdot dP =$$

$$= \int_0^{2\pi} \left(\cos t \sin t\,(-\sin t) + \cos^2 t \cdot \cos t + \sin t \cdot 0 \cdot 0 \right) dt$$

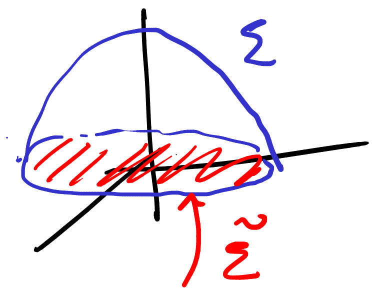
$$= \int_0^{2\pi} \left(-\sin^2 t \cos t + (1-\sin^2 t)\cos t \right) dt$$

$$= \int_0^{2\pi} \Big(\underbrace{\cos t}_{\text{contributo} = 0} - 2\sin^2 t \cos t \Big)\, dt = 0$$

$$\Big[\ldots \sin^3 t \Big]_0^{2\pi} = 0$$

Nell'esempio:

$$\Phi_\Sigma(\text{rot } F) = \Phi_{\tilde\Sigma}(\text{rot } F)$$

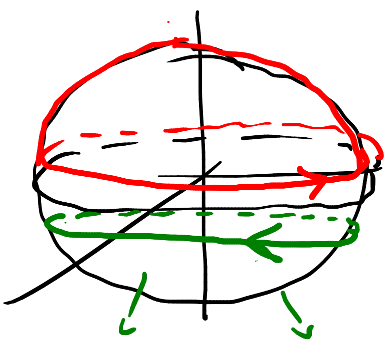


$\tilde\Sigma$: $\sigma(u,v) = (u,v,0)$

$N_\sigma(u,v) = (0,0,1)$

$(u,v) \in \bar B_1(0,0)$

$$\Phi_{\tilde\Sigma}(\text{rot} F) = \iint_{\bar B_1(0,0)} \underbrace{(uv, u^2, 0)\cdot(0,0,1)}_{=0}\, du\, dv = 0$$



Esempi (sulla divergenza)

- $F(x,y,z) = (xy, x^2, yz)$ $\quad \mathbb{R}^3$

  $\operatorname{div} F(x,y,z) = y + 0 + y = 2y$

- $F(x,y,z) = (x,y,z)$ $\quad \mathbb{R}^3$

  $\operatorname{div} F(x,y,z) = 1+1+1 = 3$

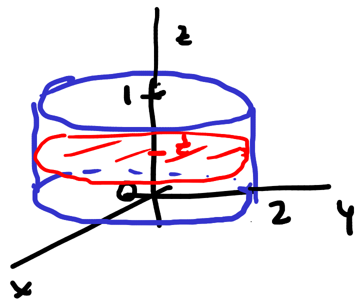
- $F(x,y,z) = (y,-x,0)$ $\quad \mathbb{R}^3$

  $\operatorname{div} F(x,y,z) = 0+0+0 = 0$

Es. (teor. della divergenza)

$F(x,y,z) = (xy^2, x^2y, (x^2+y^2)z^2)$

$T = \{(x,y,z) \mid x^2+y^2 \le 4, \ 0 \le z \le 1\}$

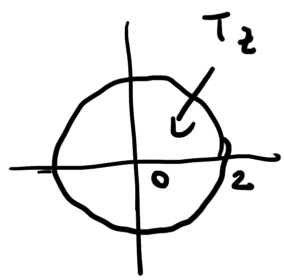


1^o membro:

$\operatorname{div} F(x,y,z) = y^2 + x^2 + 2z(x^2+y^2) = (x^2+y^2)(1+2z)$

$$\iiint_T \operatorname{div} F(x,y,z)\,dx\,dy\,dz = \iiint_T (x^2+y^2)(1+2z)\,dx\,dy\,dz$$

$$\overset{\text{per strati}}{=} \int_0^1 \left(\iint_{T_z} (x^2+y^2)(1+2z)\,dx\,dy \right) dz$$



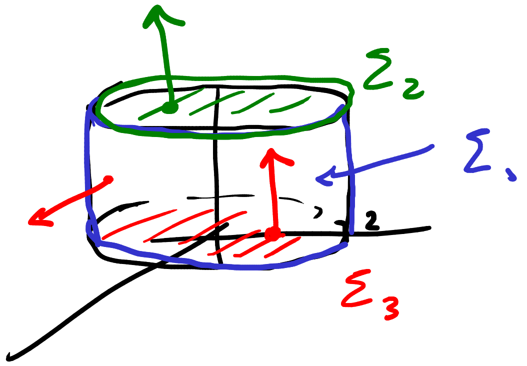
$$= \int_0^1 (1+2z) \left(\underbrace{\iint_{T_z} (x^2+y^2)\,dx\,dy}_{T_z = \overline{B}_2 \text{ non dip. da } z} \right) dz$$

$$= \int_0^1 (1+2z)\,dz \cdot \iint_{\bar{B}_2(0,0)} (x^2+y^2)\,dx\,dy$$

$$= \left[z+z^2\right]_0^1 \cdot \int_0^2 \rho^3\,d\rho \cdot \int_0^{2\pi} d\theta = 2 \cdot 4 \cdot 2\pi$$

$$= 16\pi$$

2° membro:



Σ_1 : $\sigma(\theta, z) = (2\cos\theta,\ 2\sin\theta,\ z)$

$(\theta, z) \in [0, 2\pi] \times [0,1] =: K$

So che: $N_\sigma(\theta, z) = (2\cos\theta,\ 2\sin\theta,\ 0)$ punta verso l'esterno

$F(x,y,z) = (xy^2,\ x^2y,\ (x^2+y^2)z^2)$

$$\Phi_{\Sigma_1}(F) =$$

$$\iint_K (2\cos\theta\, 4\sin^2\theta,\ 4\cos^2\theta\, 2\sin\theta,\ 4z^2) \cdot (2\cos\theta,\ 2\sin\theta, 0)\,d\theta\,dz$$

$$= \iint_K (16\cos^2\theta\sin^2\theta + 16\cos^2\theta\sin^2\theta + 0)\,d\theta\,dz$$

$$= \int_0^{2\pi} 32\cos^2\theta\sin^2\theta\,d\theta \cdot \int_0^1 dz$$

$$= \int_0^{2\pi} 8\cdot(2\cos\theta\sin\theta)^2\,d\theta \cdot 1$$

$$= \int_0^{2\pi} 8\cdot\sin^2(2\theta)\,d\theta = \int_0^{2\pi} 8\,\frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= 8\pi$$

$F(x,y,z) = (xy^2, x^2y, (x^2+y^2)z^2)$

Σ_2: $\sigma(u,v) = (u,v,1)$ $\quad (u,v) \in \bar{B}_2(0,0)$

$N_\sigma(u,v) = (0,0,1)$

$$\Phi_{\Sigma_2}(F) = \iint_{\bar{B}_2(0,0)} (uv^2, u^2v, u^2+v^2) \cdot (0,0,1)\, du\, dv$$

$$= \iint_{\bar{B}_2(0,0)} (u^2+v^2)\, du\, dv \underset{\substack{\uparrow \\ \text{coord.} \\ \text{pol.}}}{=} \int_0^2 \rho^3 d\rho \cdot \int_0^{2\pi} d\theta$$

$$= 4 \cdot 2\pi = 8\pi$$

Σ_3: $\sigma(u,v) = (u,v,0)$ $\quad (u,v) \in \bar{B}_2(0,0)$

$N_\sigma(u,v) = (0,0,1)$ PUNTA VERSO L'INTERNO DI T !!

$$\Phi_{\Sigma_3}(F) = -\iint_{\bar{B}_2(0,0)} (uv^2, u^2v, 0) \cdot (0,0,1)\, du\, dv$$

$$= -0 = 0$$

$$\underset{\substack{\shortparallel \\ \partial T^+}}{\Phi_\Sigma(F)} = \Phi_{\Sigma_1}(F) + \Phi_{\Sigma_2}(F) + \Phi_{\Sigma_3}(F)$$

$$= 8\pi + 8\pi + 0 = 16\pi$$

$F(x,y) = (y, 2x)$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2 - 1 \equiv 1$$

Es. (asteroide)

$$r(t) = (\cos^3 t,\ \sin^3 t) \qquad t \in [0, 2\pi]$$

$$r'(t) = (\ 3\cos^2 t\,(-\sin t),\ \ 3\sin^2 t \cos t\,)$$

$$= \underbrace{3\cos t \sin t}_{\neq 0}\ (\underbrace{-\cos t,\ \sin t}_{\| \ \| = 1\ \neq 0})$$

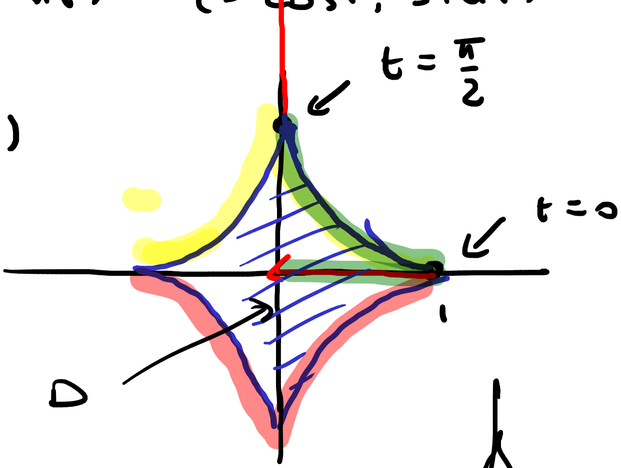
$$t \notin \{0, \tfrac{\pi}{2}, \pi, \tfrac{3}{2}\pi, 2\pi\}$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{3\cos t \sin t\ (-\cos t, \sin t)}{3\,|\cos t|\,|\sin t|}$$

$$= \operatorname{sign}(\cos t)\ \operatorname{sign}(\sin t)\ (-\cos t, \sin t)$$

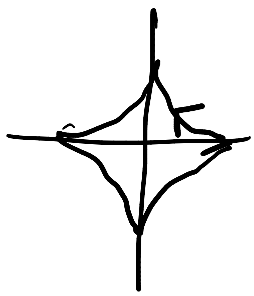
$t \to 0^+$: $T(t) \to 1 \cdot 1 \cdot (-1, 0)$

$t \to \frac{\pi}{2}^-$: $T(t) \to 1 \cdot 1 \cdot (0, 1)$



$$F(x,y) = \left(-\frac{y}{2}, \frac{x}{2}\right)$$

$$m_2(D) = \int_{\partial D^+} F(P) \cdot dP$$



$$= \int_0^{2\pi} \left(-\frac{\sin^3 t}{2}\ 3\cos^2 t(-\sin t) + \frac{\cos^3 t}{2}\ 3\sin^2 t \cos t \right) dt$$

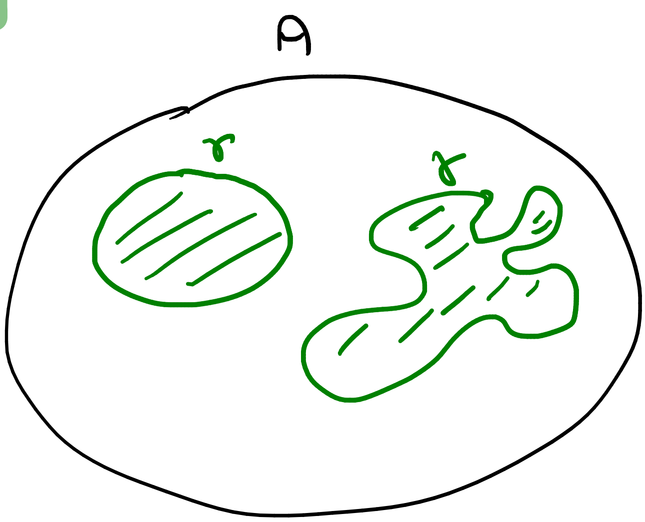
$$= \int_0^{2\pi} \left(\frac{3}{2} \sin^4 t \cos^2 t + \frac{3}{2} \sin^2 t \cos^4 t \right) dt$$

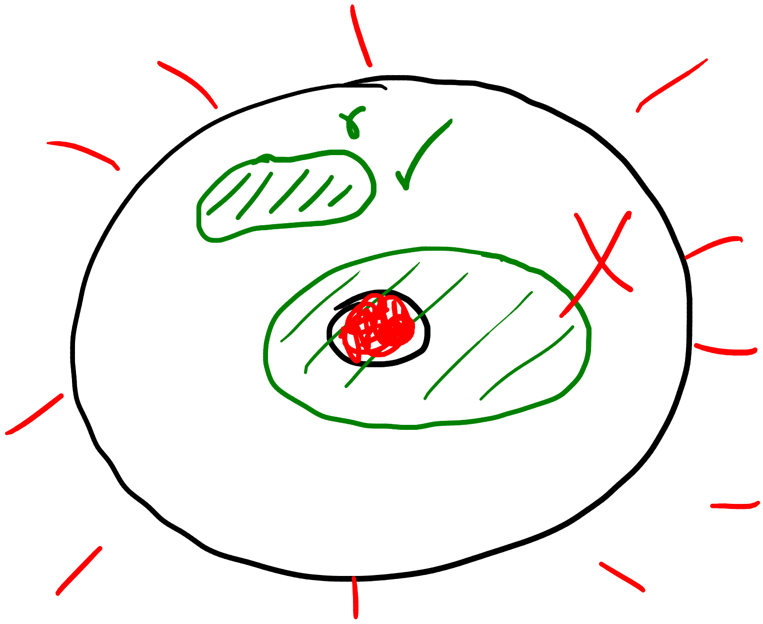
$$= \int_0^{2\pi} \frac{3}{2} \sin^2 t \cos^2 t \left(\sin^2 t + \cos^2 t \right) dt$$

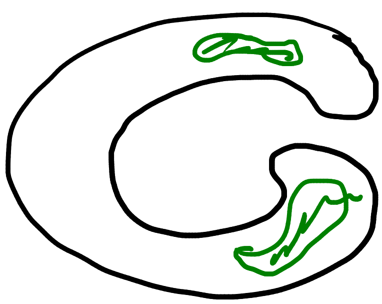
$$= \int_0^{2\pi} \frac{3}{2} \sin^2 t \cos^2 t \, dt = \int_0^{2\pi} \frac{3}{8} (\underbrace{2 \sin t \cos t}_{=\sin 2t})^2 dt$$

$$= \int_0^{2\pi} \frac{3}{8} \, \frac{1 - \overbrace{\cos(4t)}^{\text{contributo } = 0}}{2} \, dt = \frac{3}{8} \cdot \frac{1}{2} \cdot 2\pi = \frac{3}{8}\pi.$$

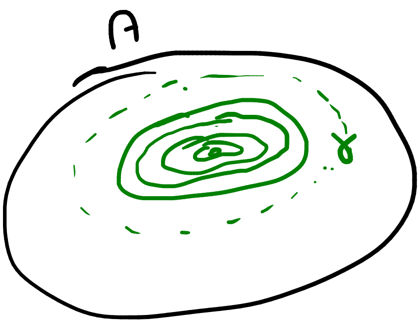
Esempi (insiemi semplicemente connessi)

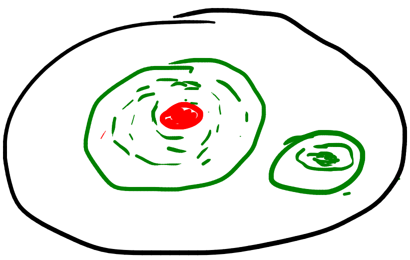


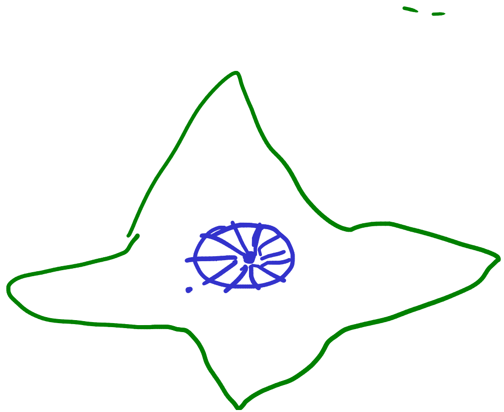




- no stellato
- sempl. conn.



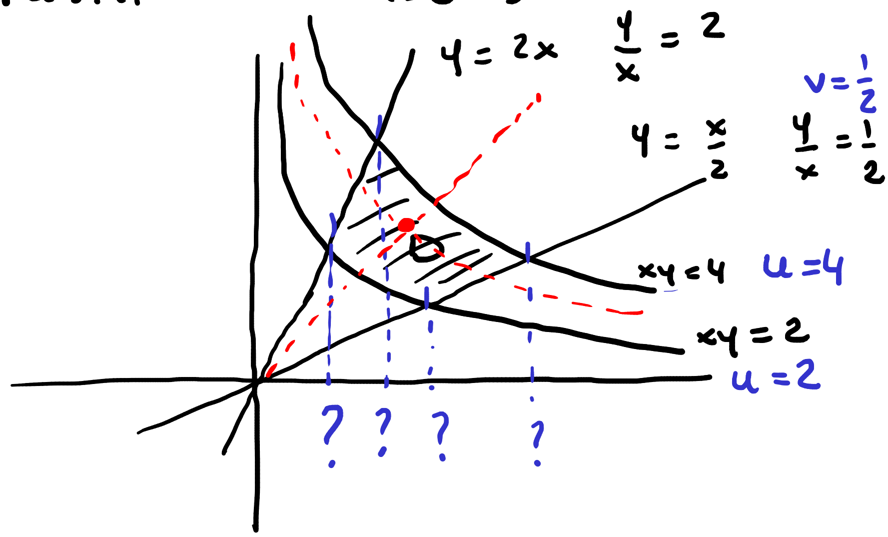




Esempio (camb. variabili "ad hoc")

$$\iint_D x^2 y^2 \, dx\, dy$$

$$\left.\begin{aligned} u &= xy \\ v &= \frac{y}{x} \end{aligned}\right\} \quad \Phi^{-1}$$



$$\tilde{D} = \{(u,v) \mid 2 \le u \le 4, \ \tfrac{1}{2} \le v \le 2\}$$

Ricavo x, y in funzione di u, v,

tenendo presente che $(x,y) \in (0,+\infty) \times (0,+\infty)$

$$\begin{cases} y = v x \\ u = v x^2 \end{cases} \qquad x^2 = \frac{u}{v} \qquad u, v > 0 \implies \frac{u}{v} > 0$$

$$\Phi : \begin{cases} x = \sqrt{\dfrac{u}{v}} = u^{\frac{1}{2}} v^{-\frac{1}{2}} \\ y = u^{\frac{1}{2}} v^{\frac{1}{2}} \end{cases}$$

$$J_\phi(u,v) = \begin{pmatrix} \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} & -\frac{1}{2} u^{\frac{1}{2}} v^{-\frac{3}{2}} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{pmatrix}$$

$$\det J_\phi(u,v) = \frac{1}{4} v^{-1} + \frac{1}{4} v^{-1} = \frac{1}{2v}$$

$$\iint_D \overbrace{x^2 y^2}^{f(x,y)} \, dx\, dy = \iint_{\tilde{D}} f(\Phi(u,v)) \, |\det J_\phi(u,v)| \, du\, dv$$

$$= \iint_{[2,4]\times\left[\frac{1}{2},2\right]} u^2 \frac{1}{2v} \, du\, dv$$

$$= \int_2^4 u^2 \, du \cdot \int_{\frac{1}{2}}^2 \frac{1}{2v} \, dv = \ldots$$