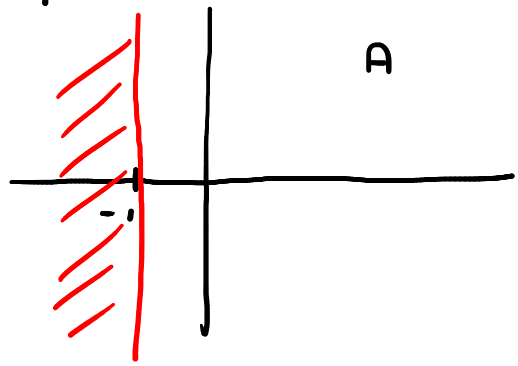
4 giugno 2024

## Esempi (forme diff.)

- $\omega(x,y) = \frac{1+y}{1+x}\,dx + \ln(1+x)\,dy$

$A = \{(x,y) \mid 1+x>0\}$




A aperto

A semipiano $\Rightarrow$ A convesso

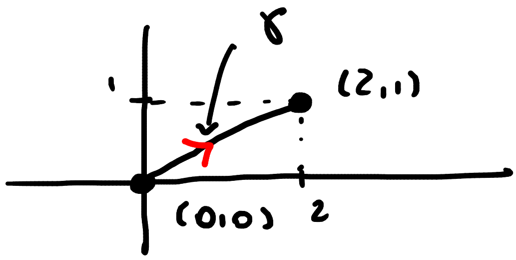
$\Rightarrow$ A stellato

$\omega$ di classe $C^1$

$$\left.\begin{array}{l} \dfrac{\partial}{\partial y}\left(\dfrac{1+y}{1+x}\right) = \dfrac{1}{1+x} \\ \qquad\qquad\qquad\qquad\quad \shortparallel \\ \dfrac{\partial}{\partial x}\left(\ln(1+x)\right) = \dfrac{1}{1+x} \end{array}\right| \Rightarrow \omega \text{ è chiusa in } A$$

Teor. di Poincaré $\Rightarrow$ $\omega$ è esatta in A.

Calcolare $\int_\gamma \omega$ con $\gamma$:




Possiamo procedere in tre modi, equivalenti ma alternativi:

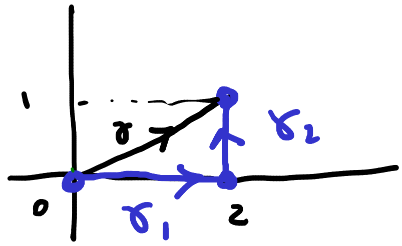
① parametrizzo $\gamma$:

$$r(t) = (0,0) + t(2,1), \quad t \in [0,1]$$

cioè: $\quad r(t) = (2t, t) \quad t \in [0,1]$

$$\int_\gamma \omega = \int_0^1 \left(\frac{1+t}{1+2t}\cdot 2 + \ln(1+2t)\cdot 1\right) dt = \dots$$

(2)




integro su una curva con gli stessi estremi, concatenamento di segmenti paralleli agli assi

Per esempio: $\tilde{\gamma}$ : concatenam. di $\gamma_1$ e $\gamma_2$

$$\int_\gamma \omega \underset{\substack{\uparrow \\ \omega \text{ esatta}}}{=} \int_{\tilde{\gamma}} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

$\gamma_1$ : $\quad r(t) = (t, \underset{\text{cost.}}{0}) \qquad t \in [0, 2]$

$$\int_{\gamma_1} \omega = \int_0^2 \left( \frac{1+0}{1+t} \cdot 1 + \cancel{\ln(1+t)} \cdot 0 \right) dt = \Big[ \ln(1+t) \Big]_0^2$$

$$= \ln(3)$$

$\gamma_2$ : $\quad r(t) = (2, t) \qquad t \in [0, 1]$

costante

$$\int_{\gamma_2} \omega = \int_0^1 \left( \cdots \cdot 0 + \ln(1+2) \cdot 1 \right) dt = \ln(3)$$

$$\Rightarrow \int_\gamma \omega = \int_{\tilde{\gamma}} \omega = 2 \ln(3)$$

$\omega(x,y) = \frac{1+y}{1+x}\, dx + \ln(1+x)\, dy$

(3) Determino una primitiva di $\omega$ :

$$f : A \to \mathbb{R} \text{ t.c. } \quad \frac{\partial f}{\partial x}(x,y) \overset{(a)}{=} \frac{1+y}{1+x}, \quad \frac{\partial f}{\partial y}(x,y) \overset{(b)}{=} \ln(1+x)$$

$$f(x,y) = (1+y) \ln(1+x) + g(y)$$

Sostituisco in (b):

$$\ln(1+x) + g'(y) = \ln(1+x) \quad \Rightarrow \quad g'(y) = 0$$

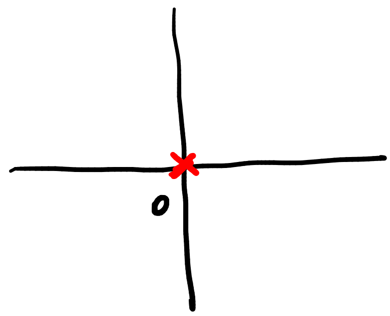
Scelgo $g(y) \equiv 0$

Quindi: $f(x,y) = (1+y)\ln(1+x)$ è primitiva di $\omega$

FFCI
$$\Rightarrow \quad \int_\gamma \omega = f(2,1) - f(0,0) = 2\ln(3) - 0 = 2\ln(3).$$

$\square$

- $$\omega(x,y) = \frac{x-y}{x^2+y^2}\,dx + \frac{x+y}{x^2+y^2}\,dy$$




$A = \mathbb{R}^2 \setminus \{(0,0)\}$

$A$ aperto, <u>non</u> stellato

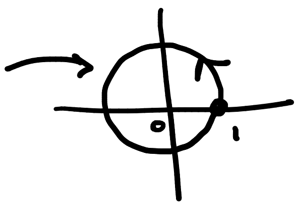
$\omega$ di classe $C^1$

$$\frac{\partial}{\partial y}\left(\frac{x-y}{x^2+y^2}\right) = \frac{-(x^2+y^2) - (x-y)2y}{(x^2+y^2)^2} = \frac{-x^2+y^2-2xy}{(x^2+y^2)^2}$$

$$\parallel$$

$$\frac{\partial}{\partial x}\left(\frac{x+y}{x^2+y^2}\right) = \frac{x^2+y^2 - (x+y)2x}{(x^2+y^2)^2} = \frac{-x^2+y^2-2xy}{(x^2+y^2)^2}$$

$\Rightarrow$ $\omega$ è chiusa in $A$.

Valuto $\int_\gamma \omega$ con $\gamma$: 

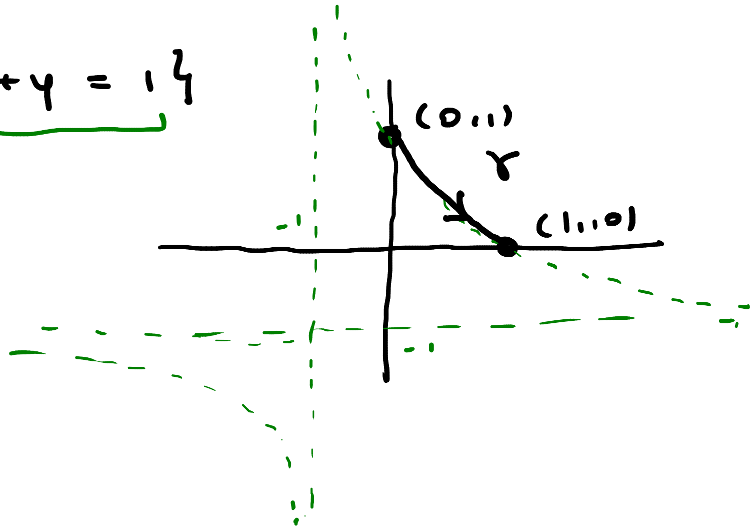

$r(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$$\int_\gamma \omega = \int_0^{2\pi} \left( \frac{\cos t - \sin t}{1}(-\sin t) + \frac{\cos t + \sin t}{1}\cos t \right) dt$$

$$= \int_0^{2\pi} \left(-\cos t \cancel{\sin t} + \sin^2 t + \cos^2 t + \cancel{\sin t} \cos t\right) dt$$

$$= 2\pi \neq 0 \quad \Rightarrow \quad \omega \text{ } \underline{\underline{\text{non}}} \text{ è esatta in } A.$$

Calcolare l'int. di $\omega$ sulla curva semplice di estremi: $(0,1)$ e $(1,0)$ e sostegno contenuto in $\{(x,y) \mid \underbrace{xy + x + y = 1}\}$




$\Leftrightarrow \quad (x+1)y = 1 - x$

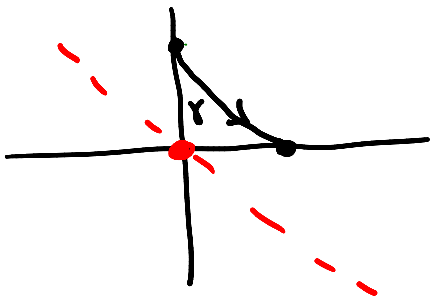
$x \neq -1$
$\Leftrightarrow \quad y = \dfrac{1-x}{1+x}$

Parametrizzo $\gamma$: $\quad \gamma(t) = \left(t, \dfrac{1-t}{1+t}\right) \quad t \in [0,1]$

$$\int_\gamma \omega = \int_0^1 \left( \frac{t - \frac{1-t}{1+t}}{t^2 + \left(\frac{1-t}{1+t}\right)^2} \cdot 1 + \frac{t + \frac{1-t}{1+t}}{t^2 + \left(\frac{1-t}{1+t}\right)^2} \cdot \frac{-(1+t)-(1-t)}{(1+t)^2} \right) dt$$

!!!

Però:



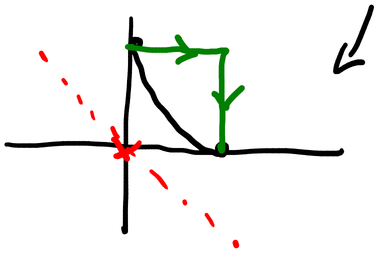

$A_+ = \{(x,y) \mid x + y > 0\}$
ap. convesso $\Rightarrow$ stellato

$\omega|_{A_+}$ è chiusa

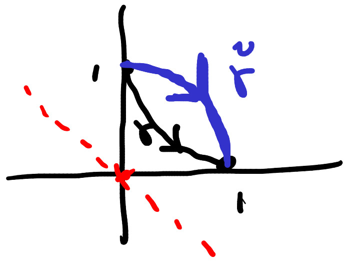
$\Rightarrow \quad \omega|_{A_+}$ è esatta

$\Rightarrow$ posso integrare su una qualsiasi curva con sostegno contenuto in $A_+$ e estremi $(0,1)$ e $(1,0)$

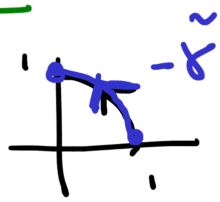
Potrei scegliere



oppure

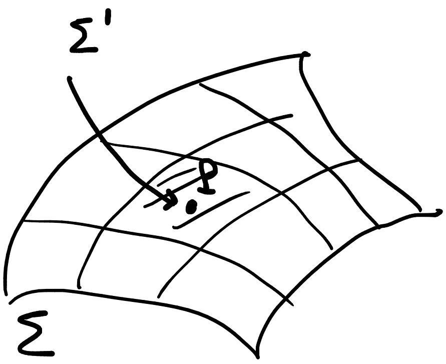



$r(t) = (\cos t, \sin t)$, $t \in [0, \frac{\pi}{2}]$




$$\int_\gamma \omega = \int_{\tilde\gamma} \omega = -\int_0^{\frac{\pi}{2}} (\dots\dots)\,dt = -\int_0^{\frac{\pi}{2}} 1\,dt = -\frac{\pi}{2}$$

Motivazione per "$\|N_\sigma(u,v)\|$" $du\,dv$ :

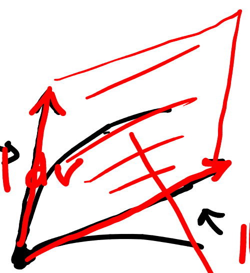



$= \sigma(u,v)$

$$\sum_{\Sigma'} f(P) \cdot \text{misura di } \Sigma'$$

??

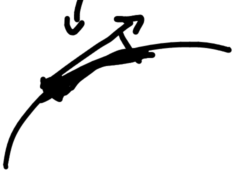

$\|\sigma_v(u,v)\|\,dv$

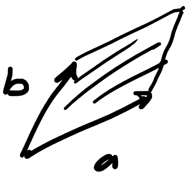


$\|\sigma_u(u,v)\|\,du$

$\sigma(u,v)$

$\|r'(t)\|\,dt$






area $= \|a \times b\|$

area $=$

$\|(\sigma_u(u,v)\,du) \times (\sigma_v(u,v)\,dv)\|$

$= \|\sigma_u(u,v) \times \sigma_v(u,v)\|\,du\,dv$

$= \| N_\sigma(u,v) \| \, du\, dv$

Esempi (int. di sup.)

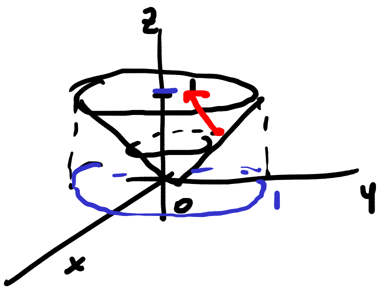
- $f(x,y,z) = z$

$\Sigma$ : $\sigma(u,v) = (u\cos v,\ u\sin v,\ u)$

$(u,v) \in [0,1] \times [0, 2\pi] =: K$

Oss: $x = u\cos v$, $y = u\sin v$, $z = u$ $\Big| \Rightarrow x^2+y^2 = u^2$

$\Rightarrow \underline{\sqrt{x^2+y^2}} = u = \underline{z}$ sup. conica




$\sigma \in C^1(\mathbb{R}^2, \mathbb{R}^3)$ ✓

$\sigma(u,v) = (u\cos v,\ u\sin v,\ u)$

$\sigma_u(u,v) = (\cos v,\ \sin v,\ 1)$

$\sigma_v(u,v) = (-u\sin v,\ u\cos v,\ 0)$

$N_\sigma(u,v) = (-u\cos v,\ -u\sin v,\ u)$

$\| N_\sigma(u,v) \| = \sqrt{u^2\cos^2 v + u^2 \sin^2 v + u^2} = \sqrt{2}\, u$

Oss: $(u,v) \in \overset{\circ}{K} \Rightarrow u \neq 0 \Rightarrow \| N_\sigma(u,v) \| \neq 0$

$\Rightarrow$ la sup. è regolare

$$\int_\Sigma f \, dS \overset{\text{def}}{=} \iint_K f(\sigma(u,v)) \, \| N_\sigma(u,v) \| \, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} u \sqrt{2} u \, du\, dv = \sqrt{2} \int_0^1 u^2 du \int_0^{2\pi} dv$$

$= \dots$

- $\sigma(u,v) = (u\cos v,\ u\sin v,\ \underbrace{v}_{\text{iniettiva}})$

$(u,v) \in [0,1] \times [0,2\pi] =: \underbrace{K}_{\text{dom. regolare}}$

$\sigma \in C^1(\mathbb{R}^2, \mathbb{R}^3)$ ✓

$\sigma_u(u,v) = (\cos v,\ \sin v,\ 0)$

$\sigma_v(u,v) = (-u\sin v,\ u\cos v,\ 1)$

$N_\sigma(u,v) = (\sin v,\ -\cos v,\ u)$

$\|N_\sigma(u,v)\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2} \neq 0$

$\underbrace{\forall (u,v) \in K \text{ (non solo in } \mathring{K})}$

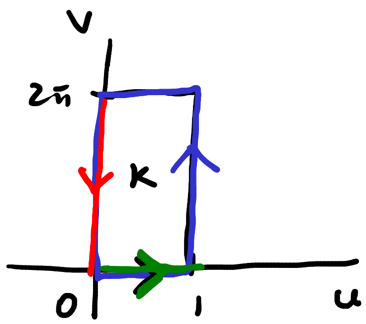
Oss: $\sigma$ è sup. regolare con bordo

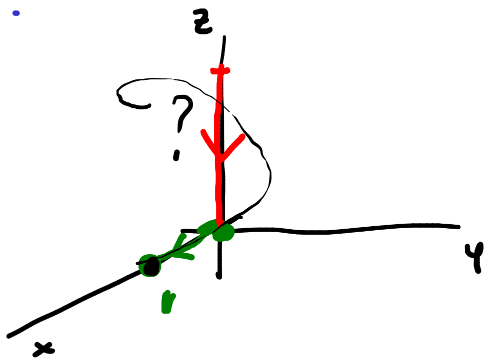
$$\int_\Sigma f\, dS = \iint_{[0,1]\times[0,2\pi]} v\sqrt{1+u^2}\, du\, dv = \int_0^1 \sqrt{1+u^2}\, du \cdot \int_0^{2\pi} v\, dv$$

$= \ldots$

$\sqrt{1+u^2} = u + t$

....




$\sigma(u,v) = (u\cos v,\ u\sin v,\ v)$

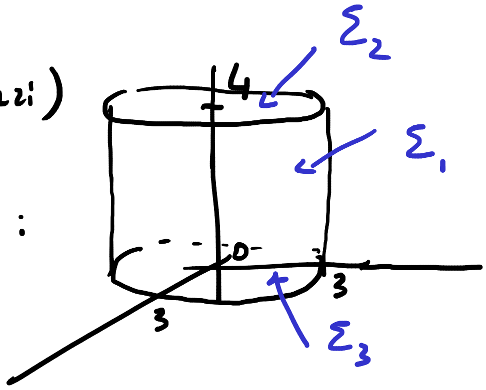
$\sigma(u,0) = (u, 0, 0) \qquad u \in [0,1]$

$\sigma(0,v) = (0, 0, v)$

$\sigma(u,v) = (\cos u, \sin v, v) \qquad v \in [0, 2\pi]$

**Esempio** (int. su sup. regolare a pezzi)

$f(x,y,z) = z, \qquad \int_{\partial T} f \, dS \qquad T:$




$\partial T$ = unione di $\Sigma_1, \Sigma_2, \Sigma_3$

$$\Rightarrow \int_{\partial T} f dS = \int_{\Sigma_1} f dS + \int_{\Sigma_2} f dS + \int_{\Sigma_3} f dS$$

$\Sigma_1$: $\quad \sigma(t,z) = (3\cos t, 3\sin t, z)$

$(t,z) \in [0, 2\pi] \times [0,4] =: K$

$\sigma_t(t,z) = (-3\sin t, 3\cos t, 0)$

$\sigma_z(t,z) = (0, 0, 1)$ irrilevante per questo esempio

$N_\sigma(t,z) = (3\cos t, 3\sin t, 0)$ (nota: punta verso l'esterno di $T$)

$\|N_\sigma(t,z)\| = \dots = 3$

$$\int_{\Sigma_1} f dS = \iint_{[0,2\pi]\times[0,4]} z \cdot 3 \, dt \, dz = \int_0^{2\pi} 3 \, dt \cdot \int_0^4 z \, dz$$

$$= 6\pi \cdot 8 = \underline{48\pi}$$

$\Sigma_2$: parametrizzo come sup. grafico

$\sigma(u,v) = (u, v, \overset{g(u,v)}{4}) \qquad (u,v) \in \bar{B}_3(0,0)$

$N_\sigma(u,v) = (-g_u(u,v), -g_v(u,v), 1)$

$$= (0,0,1)$$

(Nota: punta verso l'esterno di $T$)

$$\|N_\sigma(u,v)\| = 1$$

$$\int_{\Sigma_2} f\,dS = \iint_{\bar{B}_3(0,0)} 4\cdot 1\,du\,dv = 4\cdot 9\pi = \underline{36\pi}$$

$$\Sigma_3 : \quad \sigma(u,v) = (u,v,\underset{=}{0}) \qquad (u,v)\in \bar{B}_3(0,0)$$

$$N_\sigma(u,v) = (0,0,1)$$

(Nota: punta verso l'interno di $T$)

$$\int_{\Sigma_3} f\,dS = \iint_{\bar{B}_3(0,0)} 0\cdot 1\,du\,dv = 0$$

Conclusione: $\displaystyle\int_{\partial T} f\,dS = 48\pi + 36\pi + 0 = \dots$

Esempi (area di superfici)

- sup. sferica di raggio $r$

$$\sigma(\varphi,\theta) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$$

$$(\varphi,\theta)\in[0,\pi]\times[0,2\pi] =: K$$

$$\sigma\in C^1(\mathbb{R}^2,\mathbb{R}^3)$$

$$\sigma_\varphi(\varphi,\theta) = (r\cos\varphi\cos\theta,\ r\cos\varphi\sin\theta,\ -r\sin\varphi)$$

$$\sigma_\theta(\varphi,\theta) = (-r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta,\ 0)$$

$$N_\sigma(\varphi,\theta) = (r^2\sin^2\varphi\cos\theta,\ r^2\sin^2\varphi\sin\theta,\ r^2\sin\varphi\cos\varphi)$$

$$= r\sin\varphi\ \sigma(\varphi,\theta)$$

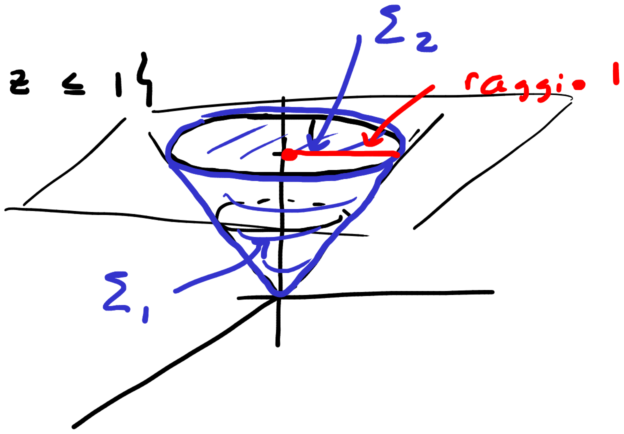
$\|N_\sigma(\varphi,\theta)\| = r\sin\varphi \cdot r = r^2 \sin\varphi$ ($\neq 0$ in $\mathring{K}$)

$$A(\Sigma) = \iint_{[0,\pi]\times[0,2\pi]} r^2 \sin\varphi \, d\varphi \, d\theta$$

$$= r^2 \int_0^\pi \sin\varphi \, d\varphi \int_0^{2\pi} d\theta = r^2 \cdot 2 \cdot 2\pi = \underline{4\pi r^2}$$

- $T = \{(x,y,z) \mid \sqrt{x^2+y^2} \le z \le 1\}$

$A(\partial T) = ?$




$A(\partial T) = A(\Sigma_1) + A(\Sigma_2)$

$\Sigma_1$: $\sigma(u,v) = (u\cos v, u\sin v, u)$

$(u,v) \in [0,1] \times [0,2\pi] =: K$

già calcolato : $\|N_\sigma(u,v)\| = \sqrt{2}u$

$$A(\Sigma_1) = \iint_{[0,1]\times[0,2\pi]} \sqrt{2}u \, du \, dv = \sqrt{2}\int_0^1 u \, du \int_0^{2\pi} dv$$

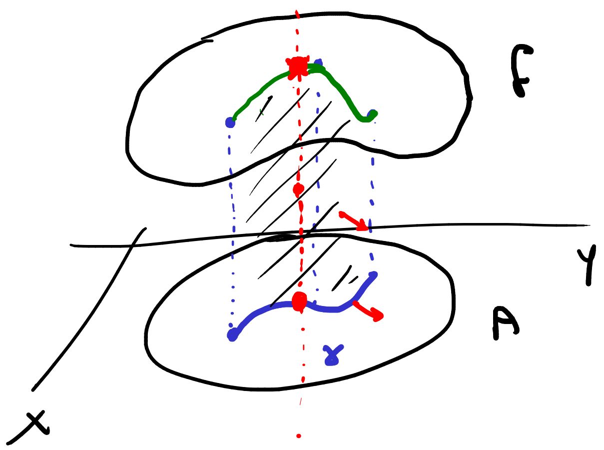
$$= \sqrt{2} \, \frac{1}{2} \cdot 2\pi = \underline{\sqrt{2}\pi}$$

$\Sigma_2$: traslato del disco di centro $(0,0)$ e raggio 1

$\Rightarrow A(\Sigma_2) = \pi \cdot 1^2 = \underline{\pi}$

$\Rightarrow A(\partial T) = \sqrt{2}\pi + \pi$ $\square$

Interpr. geom. int. curvilineo di funz. scalare




$\gamma: \quad r(t) = (x(t), y(t))$

$t \in [a,b]$

$\sigma(t,z) = (x(t), y(t), z)$ con $0 \le z \le f(x(t), y(t))$

Insieme dei parametri:

$$K := \{(t,z) \mid a \le t \le b, \ \underbrace{0}_{=:\alpha(t)} \le z \le \underbrace{f(r(t))}_{=:\beta(t)} \}$$

continua

insieme normale rispetto all' asse $t$

$\sigma$ di classe $C^1$

$\sigma_t(t,z) = (x'(t), y'(t), 0)$

$\sigma_z(t,z) = (0, 0, 1)$

$N_\sigma(t,z) = (y'(t), -x'(t), 0)$

$\|N_\sigma(t,z)\| = \sqrt{y'(t)^2 + x'(t)^2} = \|r'(t)\| \neq 0$

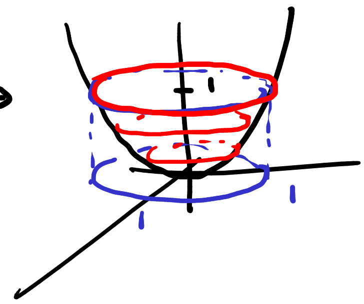
perché $(\gamma, r)$ regolare

$\Rightarrow$ sup. regolare

$$A(\Sigma) = \iint_K \|r'(t)\| \, dt = \int_a^b \left( \int_0^{f(r(t))} \|r'(t)\| \, dz \right) dt$$

$$= \int_a^b \|r'(t)\| f(r(t)) \, dt \overset{def}{=} \int_\gamma f \, ds$$

- Esempio: area di $\rightarrow$




$f(x,y) = x^2+y^2$

$(x,y) \in \overline{B}_1(0,0)$

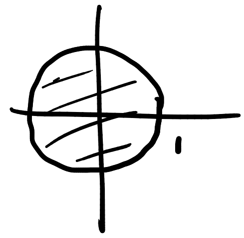
$\sigma(u,v) = (u, v, u^2+v^2) \qquad (u,v) \in \overline{B}_1(0,0)$

$\vdots$

$N_\sigma(u,v) = (-2u, -2v, 1)$

$\| N_\sigma(u,v) \| = \sqrt{4u^2+4v^2+1}$

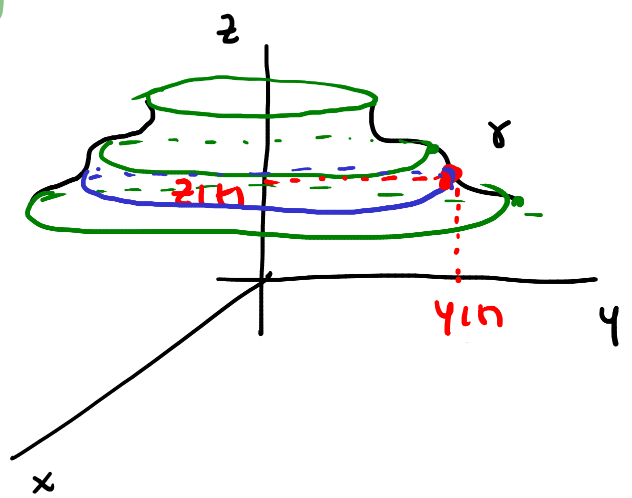
$$A(\Sigma) = \iint_{\overline{B}_1(0,0)} \sqrt{4u^2+4v^2+1}\, du\, dv$$

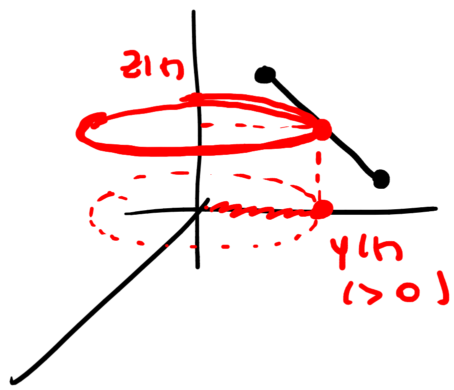



$$= \iint_{[0,1]\times[0,2\pi]} \sqrt{4\rho^2+1}\ \rho\, d\rho\, d\theta$$

$$= \frac{2\pi}{8} \int_0^1 (4\rho^2+1)^{1/2} (8\rho)\, d\rho = \dots$$

Superfici di rotazione

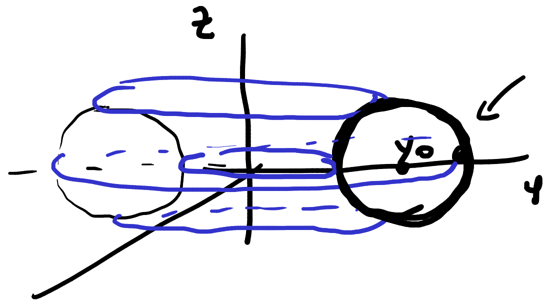



$\sigma(t,\theta) = (y(t)\cos\theta, y(t)\sin\theta, z(t))$

$(t,\theta) \in [a,b] \times [0,2\pi] =: K$

Es.




raggio $R$, centro $(y_0, 0)$

$$\gamma: \quad r(t) = (0, \; y_0 + R\cos t, \; R\sin t)$$

$$t \in [0, 2\pi]$$

$$r'(t) = (0, -R\sin t, R\cos t)$$

$$\| r'(t) \| = R$$

Usando la formula:

$$A(\Sigma) = 2\pi \int_0^{2\pi} (y_0 + R\cos t)\, R \, dt$$

$$= 2\pi \left( \int_0^{2\pi} y_0 R \, dt + \underbrace{\int_0^{2\pi} R^2 \cos t \, dt}_{=0} \right)$$

$$= 2\pi \; y_0 R \; 2\pi$$

$$= \underbrace{2\pi R}_{L(\gamma)} \cdot \underbrace{2\pi y_0}_{\text{lunghezza della circonferenza descritta da } y_0}$$

(caso particolare del teor. di Pappo - Guldino)

## Esempi (flussi)

- $F(x, y, z) = (0, 0, z)$

$$\Sigma: \quad \sigma(u, v) = (u\cos v, \; u\sin v, \; v)$$

$$(u, v) \in [0, 1] \times [0, 2\pi] =: K$$

Già osservato: sup. con bordo
$\Rightarrow$ orientabile ✓

Già calcolato: $N_\sigma(u,v) = \underline{(\sin v, -\cos v, u)}$

Calcolo $F(\sigma(u,v)) = F(u\cos v, u\sin v, v) =$

$\underline{(0, 0, v)}$

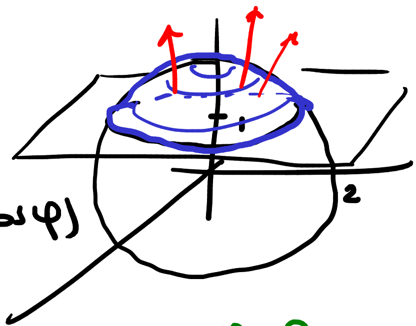
$$\Rightarrow \Phi_\Sigma(F) = \iint_K F(\sigma(u,v)) \cdot N_\sigma(u,v)\, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} (0,0,v)\cdot(\sin v, -\cos v, u)\, du\, dv$$

$$= \iint_{[0,1]\times[0,2\pi]} (0 + 0 + uv)\, du\, dv = \int_0^1 u\, du \int_0^{2\pi} v\, dv$$

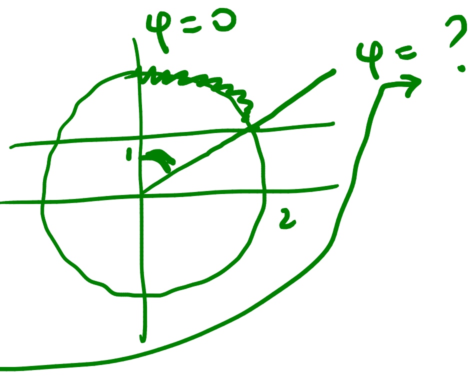
$$= \dots$$

- $F(x,y,z) = (x, y, 0)$ $\qquad \Sigma$:




$\sigma(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$

$(\varphi,\theta) \in [0, \frac{\pi}{3}] \times [0, 2\pi]$




$$\begin{cases} z = 1 \\ z = 2\cos\varphi \end{cases}$$

$\cos\varphi = \frac{1}{2}$

$\varphi = \frac{\pi}{3}$

Calcolo

$F(\sigma(\varphi,\theta)) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 0)$

Già calcolato:

$$N_\sigma(\varphi,\theta) = (4\sin^2\varphi\cos\theta,\ 4\sin^2\varphi\sin\theta,\ 4\sin\varphi\cos\varphi)$$

$$F(\sigma(\varphi,\theta)) \cdot N_\sigma(\varphi,\theta) = 8\sin^3\varphi\cos^2\theta + 8\sin^3\varphi\sin^2\theta + 0$$

$$= 8\sin^3\varphi$$

$$\Phi_\Sigma(F) = \iint_{[0,\frac{\pi}{3}]\times[0,2\pi]} 8\sin^3\varphi \, d\varphi \, d\theta$$

$$= 8 \cdot 2\pi \int_0^{\pi/3} (1-\cos^2\varphi)\sin\varphi \, d\varphi = \cdots$$