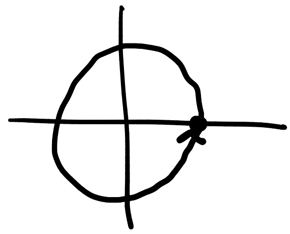
31 maggio 2024

Esempi (lunghezza)

- $r(t) = (\cos t, \sin t) \qquad t \in [0, 2\pi]$

  reg, semplice

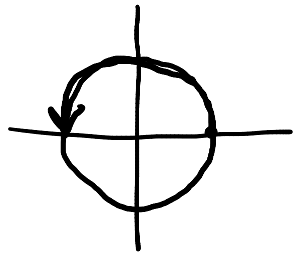
  

  $r'(t) = (-\sin t, \cos t)$

  $\Rightarrow \|r'(t)\| = 1 \quad \forall t$

  $\Rightarrow L(\gamma, r) = \int_0^{2\pi} 1 \, dt = 2\pi$

- $r(t) = (\cos t, \sin t) \qquad t \in [0, 3\pi]$

  reg, non semplice

  $L(\gamma, r) = \int_0^{3\pi} 1 \, dt = 3\pi$

- $r(t) = (t + \sin t, \underbrace{\cos t}_{\text{iniettiva}}) \qquad t \in [0, \pi]$

  regolare, semplice

  $r'(t) = (1 + \cos t, -\sin t)$

  $\|r'(t)\| = \sqrt{1 + \cos^2 t + 2\cos t + \sin^2 t} = \sqrt{2(1 + \cos t)}$

  $\Rightarrow L(\gamma, \cancel{r}) = \int_0^{\pi} \sqrt{2(1+\cos t)} \, dt = \dots$

  $\qquad \qquad \sqrt{2(1+\cos t)} = \sqrt{\cos^2 \frac{t}{2}} = \cos \frac{t}{2} \qquad \left(\frac{t}{2} \in [0, \frac{\pi}{2}] \right)$

- $r(t) = (2\cos t, 2\sin t, 3t) \qquad t \in [0, 2\pi]$

Già nota (semplice, regolare)

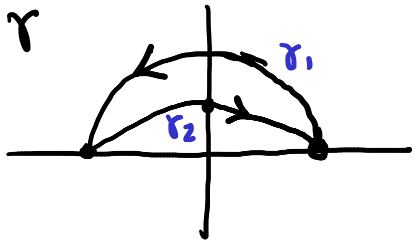
$$r'(t) = (-2\sin t,\ 2\cos t,\ 3)$$

$$\|r'(t)\| = \sqrt{4+9} = \sqrt{13}$$

$$L(\gamma) = \int_0^{2\pi} \sqrt{13}\, dt = \dots$$

- lunghezza del concatenamento di:
  - $r(t) = (3\cos t,\ 3\sin t)$ $\quad t \in [0,\pi]$
  - curva grafico associata a $f(x) = 1 - \frac{x^2}{9}$ $\quad x \in [-3,3]$

γ



$$L(\gamma) = L(\gamma_1) + L(\gamma_2)$$

$$= 3\pi + \int_{-3}^{3} \sqrt{1 + \frac{4}{81}x^2}\, dx$$

$$\sqrt{1 + \frac{4}{81}x^2} = \frac{2}{9}x + t \quad \dots$$

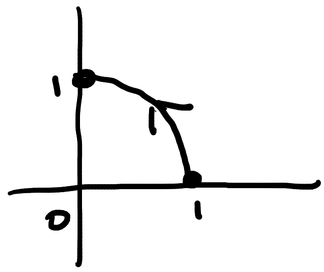
$$1 + \cancel{\frac{4}{81}x^2} = \cancel{\frac{4}{81}x^2} + \frac{4}{9}xt + t^2$$

$$x = \dots(t)$$

Esempio (integrale curvilineo di campo vettoriale)

$$F(x,y) = (y,\ xy) \qquad (x,y) \in \mathbb{R}^2$$

- $r(t) = (\cos t,\ \sin t)$ $\quad t \in [0, \frac{\pi}{2}]$



$$r'(t) = (-\sin t,\ \cos t)$$

$$F(r(t)) = F(\cos t, \sin t)$$

$$= (\sin t,\ \cos t \sin t)$$

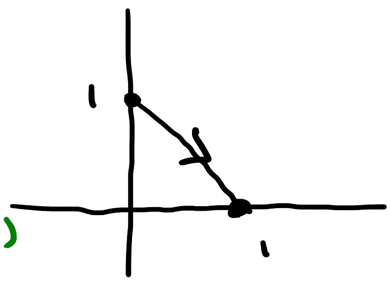
$$\Rightarrow \int_\gamma F(P)\cdot dP \overset{def}{=} \int_0^{\frac{\pi}{2}} (\sin t, \cos t \sin t)\cdot(-\sin t, \cos t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} (-\sin^2 t + \cos^2 t \sin t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\cos 2t - 1}{2} + \cos^2 t \sin t \right) dt$$

$$= \left[+ \frac{\sin 2t}{4} - \frac{t}{2} - \frac{1}{3}\cos^3 t \right]_0^{\frac{\pi}{2}} = -\frac{\pi}{4} + \frac{1}{3}$$

• $f(x) = 1 - x, \quad x \in [0,1]$



$r(t) = (t, 1-t)$

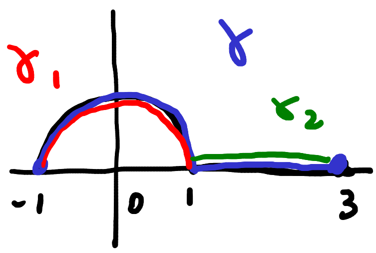
$r'(t) = (1, -1)$

$F(x,y) = (y, xy)$

$$\int_\gamma F(P)\cdot dP = \int_0^1 F(t, 1-t)\cdot r'(t)\, dt$$

$$= \int_0^1 (1-t, t(1-t))\cdot(1,-1)\, dt$$

$$= \int_0^1 (1-t - t(1-t))\, dt = \int_0^1 (1-t)^2\, dt = \dots = \frac{1}{3}$$



$$\int_\gamma F(P)\cdot dP = \int_{\gamma_1} F(P)\cdot dP + \int_{\gamma_2} F(P)\cdot dP$$

$$= \int_1^3 F(t,0)\cdot(1,0)\, dt$$

$$= -\int_0^\pi F(\cos t, \sin t)\cdot(-\sin t, \cos t)\, dt$$

Esempio (integrale di forma differenziale)

$\omega(x,y) = y\,dx + xy\,dy$

$r(t) = (\cos t, \sin t) \qquad t \in [0, \frac{\pi}{2}]$

$$\int_\gamma \omega = \int_0^{\frac{\pi}{2}} \left(\sin t\,(-\sin t) + \cos t \sin t\,(\cos t)\right) dt$$

$= \dots$ lo stesso di prima!

Es: determinare una primitiva di

$$\omega(x,y,z) = (e^y + 2xz)\,dx + \left(x e^y - \frac{1}{y-2}\right)dy + (x^2 + z)\,dz$$

$A = \{(x,y,z) \mid y \neq 2\}$ aperto, non connesso

$= A_+ \cup A_-$ $\qquad A_+ = \{(x,y,z) \mid y > 2\}$, $A_- = \{(x,y,z) \mid y < 2\}$ $\Big\}$ aperti connessi

Cerco $f : A \to \mathbb{R}$ di classe C^1 t.c.

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = e^y + 2xz & (1) \\ \frac{\partial f}{\partial y}(x,y,z) = x e^y - \frac{1}{y-2} & (2) \\ \frac{\partial f}{\partial z}(x,y,z) = x^2 + z & (3) \end{cases} \qquad \forall (x,y,z) \in A$$

se f soddisfa (1):

"costante additiva RISPETTO a x"

$$f(x,y,z) = x e^y + x^2 z + g(y,z) \qquad (*)$$

Sostituisco (*) in (2) :

$$\cancel{xe^y} + 0 + \frac{\partial g}{\partial y}(y,z) = \cancel{xe^y} - \frac{1}{y-2}$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial y}(y,z) = -\frac{1}{y-2}$$

"cost. additiva rispetto a y"

$$\Rightarrow \quad g(y,z) = -\ln|y-2| + h(z) \qquad (**)$$

Sostituisco (**) in (*) :

$$f(x,y,z) = xe^y + x^2 z - \ln|y-2| + h(z) \qquad (***)$$

Sostituisco (***) in (3) :

$$0 + \cancel{x^2} + 0 + h'(z) = \cancel{x^2} + z$$

$$\Leftrightarrow \quad h'(z) = z$$

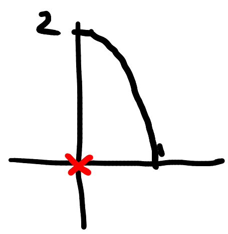
Scelgo $\quad h(z) = \dfrac{z^2}{2}$

Conclusione : <u>una</u> primitiva di ω in A è

$$f(x,y,z) = xe^y + x^2 z - \ln|y-2| + \frac{z^2}{2} \qquad \square$$

Esempi (su FFCI)

- $\omega(x,y) = \dfrac{x}{x^2+y^2}\,dx + \dfrac{y}{x^2+y^2}\,dy \qquad (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$



$$\int_\gamma \omega = ?$$

γ: $\quad \gamma(t) = (\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$

$\gamma \subset \mathbb{R}^2 \setminus \{(0,0)\}$ ✓

Conosco una primitiva di ω:

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2)$$

$$\overset{FFCI}{\Rightarrow} \quad \int_\gamma \omega = f\left(r\left(\tfrac{\pi}{2}\right)\right) - f(r(0))$$

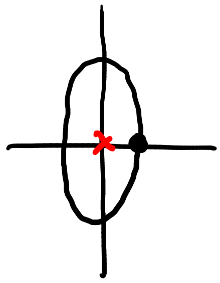
$$= f(0,2) - f(1,0) = \frac{1}{2}\ln 4 = \ln 2$$

$$\int_\gamma \omega = 0$$

γ: $r(t) = (\cos t, 2\sin t)$

$t \in [0, 2\pi]$

curva chiusa



- $\omega(x,y,z) = (e^y + 2xz)\,dx + \left(xe^y - \frac{1}{y-2}\right)dy + (x^2+z)\,dz$

  $A = \{(x,y,z) \mid y \neq 2\}$

- $r(t) = (\cos t, \underbrace{\sin t}_{\neq 2\ \forall t \in [0,2\pi]\ ?\ \text{sì!}}, t) \quad t \in [0, 2\pi]$ ✓

$$\int_\gamma \omega = f(r(2\pi)) - f(r(0))$$

$$= f(1,0,2\pi) - f(1,0,0) = (*)$$

con f primitiva di ω, già calcolata:

$$f(x,y,z) = xe^y + x^2 z - \ln|y-2| + \frac{z^2}{2}$$

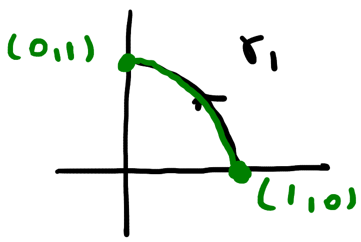
$$(*) = \cancel{1} + 2\pi - \cancel{\ln 2} + 2\pi^2 - \cancel{1} + \cancel{\ln 2}$$

• $r(t) = (\cos t, \sin t, \cos^3 t) \quad t \in [0, 2\pi]$

$$r(0) = r(2\pi) \implies \int_\gamma \omega = 0.$$

Esempio (forma diff. non esatta)

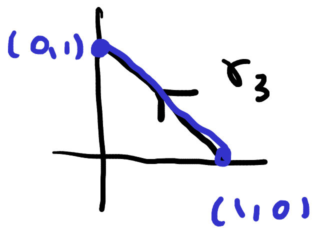
$$\omega(x,y) = y\,dx + xy\,dy$$



Già visto: $\displaystyle\int_{\gamma_1} \omega = \underbrace{-\frac{\pi}{4} + \frac{1}{3}}$

Già visto: $\displaystyle\int_{\gamma_2} \omega = \frac{1}{3}$

$\Downarrow$



$$\underbrace{\int_{\gamma_3} \omega = -\frac{1}{3}}$$

$\neq$

$\Downarrow$

ω NON è esatta!

Verifica: esatta $\implies$ chiusa

$\omega \in C^1$, $\quad \omega = df \implies \underbrace{f \in C^2}$ teor. di Schwarz

$\Updownarrow$

$$a_i = \frac{\partial f}{\partial x_i}$$

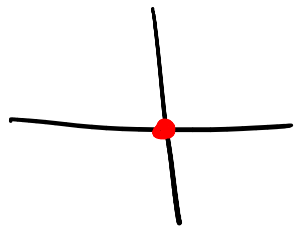
$$\underbrace{\frac{\partial a_i}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \underbrace{\frac{\partial a_j}{\partial x_i}}$$

Esempio (forma diff. chiusa e non esatta)

$$\omega(x,y) = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

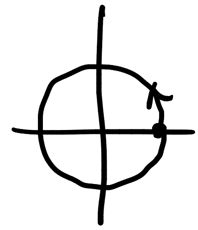
. $A = \mathbb{R}^2 \setminus \{(0,0)\}$

<u>non</u> stellato!



$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

fatelo voi



γ: $r(t) = (\cos t, \sin t)$ $\quad t \in [0, 2\pi]$

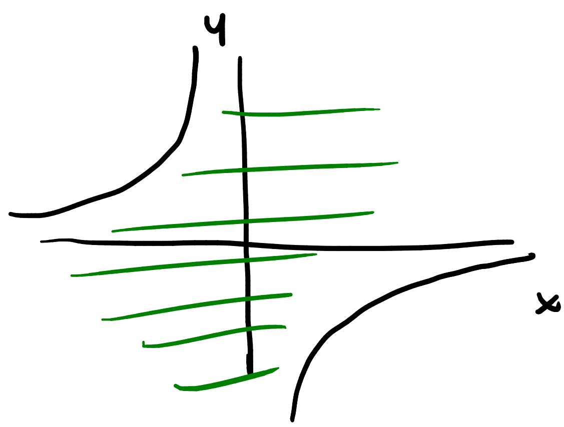
$$\int_\gamma \omega = \int_0^{2\pi} \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{1}(\cos t) \right) dt$$

$$= \int_0^{2\pi} 1\,dt = 2\pi \neq 0$$

ω non è esatta

Esempio (applicazione teor. di Poincaré)

$1 + xy > 0 \qquad xy > -1$



$A = \{(x,y,z) \mid 1 + xy > 0\}$

stellato

$$\omega(x,y,z) = \underbrace{\left(\frac{y}{1+xy} + z\right)}_{a(x,y,z)} dx + \underbrace{\frac{x}{1+xy}}_{b(x,y,z)} dy + \underbrace{x}_{c(x,y,z)} dz$$

$$\frac{\partial a}{\partial y}(x,y,z) = \frac{1+xy-yx}{(1+xy)^2} \Bigg\rangle = \quad \checkmark$$

$$\frac{\partial b}{\partial x}(x,y,z) = \frac{1+xy-xy}{(1+xy)^2}$$

$$\frac{\partial a}{\partial z}(x,y,z) = 1 \qquad \checkmark$$

$$\parallel$$

$$\frac{\partial c}{\partial x}(x,y,z) \quad 1$$

$\Rightarrow$ chiusa

$\Rightarrow$ esatta !

$$\frac{\partial b}{\partial z}(x,y,z) = 0 \qquad \checkmark$$

$$\parallel$$

$$\frac{\partial c}{\partial y}(x,y,z) = 0$$

$\square$