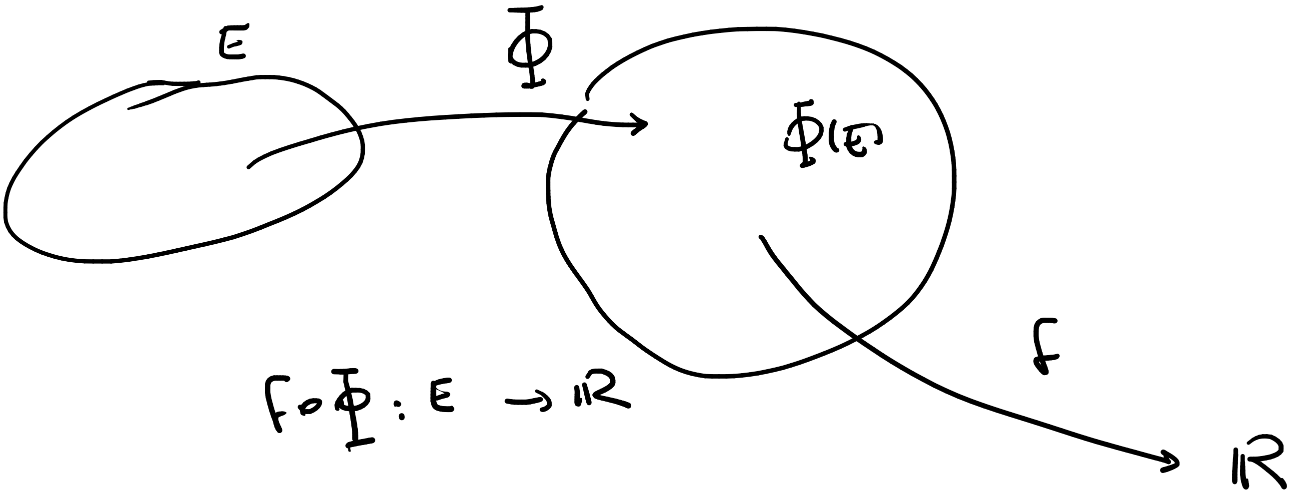
28 maggio 2024

Commenti sul teor. di cambiamento di variabili



$$f \circ \Phi : E \to \mathbb{R}$$

$f : [a,b] \to \mathbb{R}$, $\varphi : [c,d] \to [a,b]$

An. 1 :

$$\int_a^b f(x)\,dx \underset{x=\varphi(t)}{\overset{\uparrow}{=}} \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt$$

$$= \begin{cases} \varphi \text{ cresc.} & \displaystyle\int_c^d f(\varphi(t))\,|\varphi'(t)|\,dt \\ \varphi \text{ decr.} & \displaystyle\int_d^c f(\varphi(t))\,\varphi'(t)\,dt \end{cases}$$

$$= \int_c^d f(\varphi(t))\underbrace{(-\varphi'(t))}_{=|\varphi'(t)|}\,dt$$

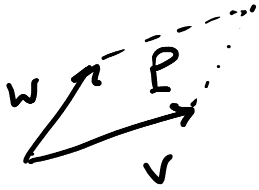
Oss: $n = 2$ $\quad x = (x_1, x_2)$

$J \in M_2$, $\det(J) \neq 0$

$L : \mathbb{R}^2 \to \mathbb{R}^2$ t.c. $L(x) = Jx$ — prod. righe per colonne

L è lineare, invertibile.

v, w lin. indip.



Ricordo: $\text{area}(P) = |\det(v \; w)|$
$\qquad\quad\; \shortparallel$
$\qquad\; m_2(P)$

L lin, inv. $\Rightarrow$ $L(v), \; L(w)$ lin. indip.
$\qquad\qquad\qquad\quad \shortparallel \qquad \shortparallel$
$\qquad\qquad\qquad\quad Jv \qquad Jw$

L(P)

$$
\begin{aligned}
m_2(L(P)) &= |\det(L(v) \; L(w))| \\
&= |\det(Jv \; Jw)| \\
&= |\det(J(v \; w))| \\
\text{Binet} \to &= |\det J| \, |\det(v \; w)| \\
&= |\det J| \, m_2(P)
\end{aligned}
$$

Oss: vale anche se L è "affine"
(lineare + costante)

$\Phi \in C^1$ Per ogni u_0:

$$\Phi(u) = \Phi(u_0) + J_\Phi(u_0)(u - u_0) + o(\|u - u_0\|)$$

vicino a u_0
$\Rightarrow$ $\Phi(u) \sim \Phi(u_0) + J_\Phi(u_0)(u - u_0)$

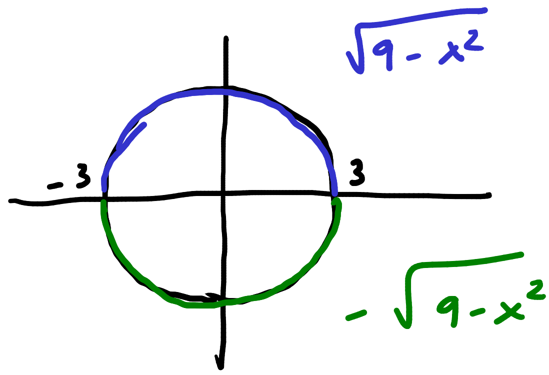
$$= \underbrace{J_\phi(u_0)\, u}_{} + \text{costante}$$

$\det \neq 0$

Esempi (coord. polari nel piano)

- $f(x,y) = x^2 - 2y^2$

  D : disco centro $(0,0)$ e raggio 3



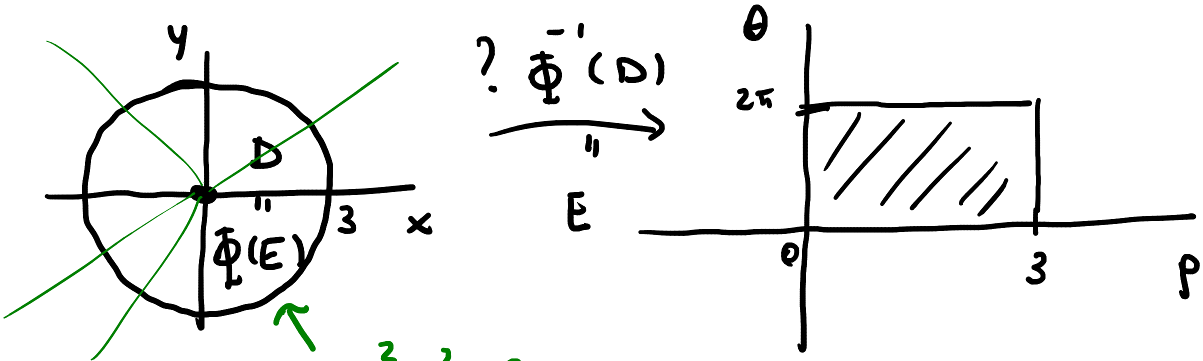
$$\iint_D f(x,y)\,dx\,dy =$$

$$\int_{-3}^{3} \left(\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2 - 2y^2)\,dy \right) dx$$

$= \dots$ (contiene $\int \sqrt{9-x^2}\,dx$ ...)

Uso coordinate polari :

$$\Phi(\rho,\theta) = (\rho\cos\theta,\ \rho\sin\theta)$$



$x^2 + y^2 = 9$

$\rho^2\cos^2\theta + \rho^2\sin^2\theta = 9 \qquad \rho^2 = 9 \qquad \rho = \cancel{\pm}\, 3$

$$E = \{ (\rho,\theta) \mid 0 \le \rho \le 3,\ 0 \le \theta \le 2\pi \}$$

$$\iint_D f(x,y)\,dx\,dy = \iint_E f(\overset{(\rho\cos\theta,\ \rho\sin\theta)}{\overset{\shortparallel}{\Phi(\rho,\theta)}})\,|\det J_\Phi(\rho,\theta)|\,d\rho\,d\theta$$

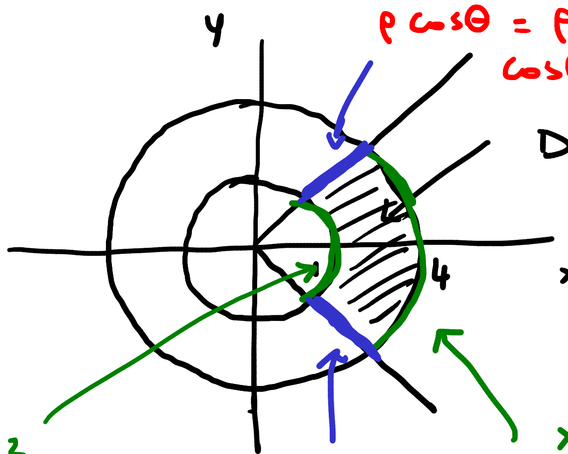
$$= \iint_{[0,3]\times[0,2\pi]} (\rho^2\cos^2\theta - 2\rho^2\sin^2\theta)\,\rho\,d\rho\,d\theta$$

$$= \iint_{[0,3]\times[0,2\pi]} \underbrace{\rho^3}_{\text{dipende solo da }\rho}(\cos^2\theta \underbrace{-2\sin^2\theta)}_{\text{dipende solo da }\theta}\,d\rho\,d\theta$$

$$= \int_0^3 \rho^3\,d\rho \cdot \int_0^{2\pi} (\cos^2\theta - 2\sin^2\theta)\,d\theta = \dots$$

- $f(x,y) = x^2 - 2y^2$

In coord. polari:



$x = y$

$\rho\cos\theta = \rho\sin\theta$

$\cos\theta = \sin\theta \quad (3,0)$

$\theta = \frac{\pi}{4}$

$x^2+y^2 = 1$

$\rho = 1$

$x = -y$

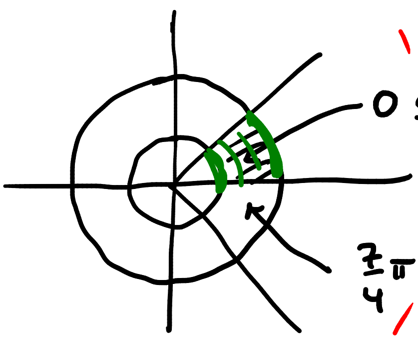
$\rho\cos\theta = -\rho\sin\theta$

$\cos\theta = -\sin\theta$

$\theta = -\frac{\pi}{4}$

$x^2+y^2 = 4$

$\rho = 2$



$\cancel{0 \le \theta \le \frac{\pi}{4}}$

$\cancel{\frac{7}{4}\pi \le \theta \le 2\pi}$

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$$\Phi^{-1}(D) =: E = \left\{ (\rho,\theta) \ \middle|\ 1 \le \rho \le 2,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \right\}$$

$$\iint_D (x^2 - 2y^2)\,dx\,dy = \iint_{[1,2]\times[-\frac{\pi}{4},\frac{\pi}{4}]} (\rho^2\cos^2\theta - 2\rho^2\sin^2\theta)\,\rho\,d\rho\,d\theta$$

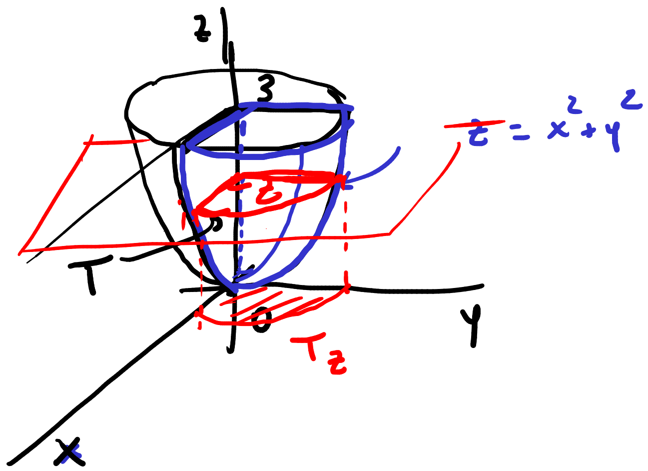
$$= \int_1^2 \rho^3\,d\rho \cdot \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos^2\theta - 2\sin^2\theta)\,d\theta = \dots$$

$$\left[\ \iint_{[a,b]\times[c,d]} g(x)h(y)\,dx\,dy = \int_a^b \left(\int_c^d g(x)h(y)\,dy\right)dx\right.$$

$$\left. = \int_a^b g(x)\left(\int_c^d h(y)\,dy\right)dx = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy\ \right]$$

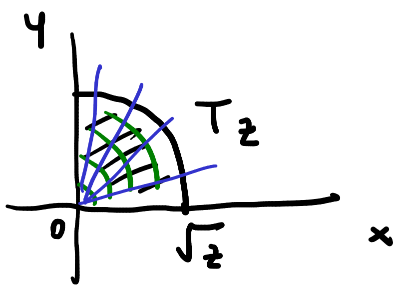
- $f(x,y,z) = xy$

? $\iiint_T f(x,y,z)\,dx\,dy\,dz$



Integro per strati:

$$I := \iiint_T f(x,y,z)\,dx\,dy\,dz = \int_0^3 \underbrace{\left(\iint_{T_z} xy\,dx\,dy\right)}_{\circledast} dz$$



Coord. polari:

$$\{(\rho,\theta)\ |\ 0 \le \rho \le \sqrt{z},\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$\circledast = \iint_{[0,\sqrt{z}]\times[0,\frac{\pi}{2}]} \rho\cos\theta\ \rho\sin\theta\ \rho\ d\rho\,d\theta$$

$$= \int_0^{\sqrt{2}} \rho^3 \, d\rho \cdot \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta = \frac{z^2}{4} \cdot \frac{1}{2} = \frac{z^2}{8}$$

$$\Rightarrow \quad I = \int_0^3 \frac{z^2}{8} \, dz = \dots$$

Se volete, ricalcolatelo per fili.

- Volume del solido delimitato da

$$z = x^2 + y^2, \quad z = 3 - 2y$$

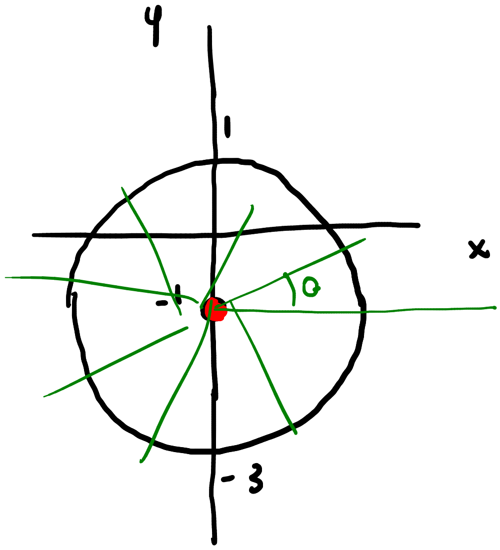
$$\text{Volume} = \iiint_T 1 \, dx\,dy\,dz$$

$$T = \{ (x,y,z) \mid (x,y) \in D, \; x^2 + y^2 \leq z \leq 3 - 2y \}$$

già calcolato

$$D = \{ (x,y) \mid x^2 + (y+1)^2 \leq 2 \}$$

$$\text{Volume} = \iint_D \left(\int_{x^2+y^2}^{3-2y} 1 \, dz \right) dx\,dy$$



$$= \iint_D (\underbrace{3 - 2y - x^2 - y^2}_{\overset{\shortparallel}{4 - x^2 - (y+1)^2}}) \, dx\,dy$$

Coord. polari di centro $(0,-1)$:

$$\Phi(\rho,\theta) = (\rho\cos\theta, \; -1 + \rho\sin\theta)$$

$$(\det J_\Phi(\rho,\theta) = \rho)$$

$$\Phi^{-1}(D) = \{ (\rho,\theta) \mid 0 \leq \rho \leq 2, \; 0 \leq \theta \leq 2\pi \}$$

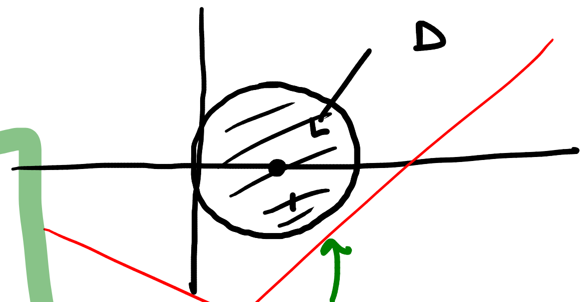
$$\text{Volume} = \iint_{[0,2]\times[0,2\pi]} (4 - \rho^2\cos^2\theta - \rho^2\sin^2\theta)\,\rho \; d\rho\, d\theta$$

$$= \iint_{[0,2]\times[0,2\pi]} (4-\rho^2)\rho\, d\rho\, d\theta$$

$$= \int_0^2 (4\rho - \rho^3)\, d\rho \cdot \int_0^{2\pi} 1 \cdot d\theta$$

$$= 2\pi \left[2\rho^2 - \frac{\rho^4}{4} \right]_0^2 = \dots$$

- $\displaystyle\iint_D \sqrt{x^2+y^2}\, dx\, dy$



coord. polari
di centro (0,0) !!

$f(\rho,\theta) = \rho$ FACILE!

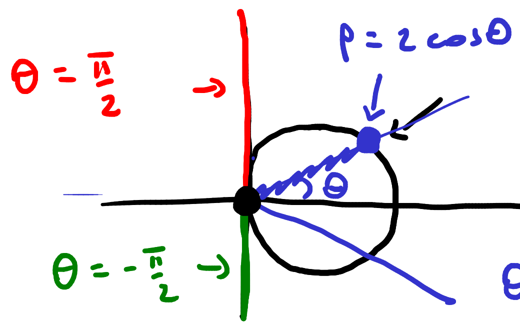
~~coord. polari di centro (1,0) !!~~

~~$\Phi^{-1}(D) = [0,1]\times[0,2\pi]$ FACILE!~~

$$f(1+\rho\cos\theta,\ \rho\sin\theta) = \sqrt{1+2\rho\cos\theta+\rho^2} \quad ?!!$$

Scelgo $\Phi(\rho,\theta) = (\rho\cos\theta,\ \rho\sin\theta)$

Come si "traduce" D ?

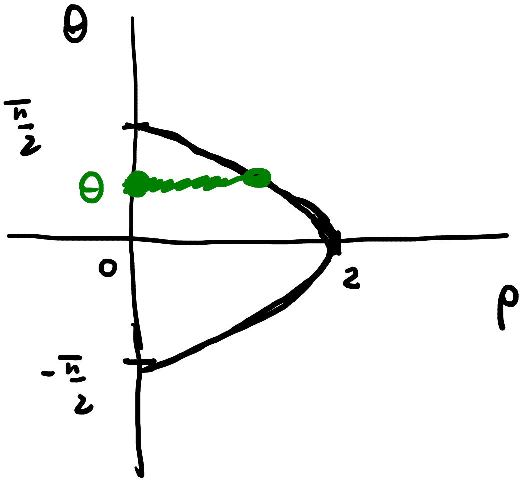


$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$(x-1)^2 + y^2 = 1$$
$$x^2 - 2x + y^2 = 0 \;\rightarrow\; \rho^2 - 2\rho\cos\theta = 0$$
$$\rho(\rho - 2\cos\theta) = 0$$

- $\rho = 2\cos\theta$

$$\Rightarrow \underbrace{\Phi^{-1}(D)}_{=:E} = \left\{ (\rho, \theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le \rho \le 2\cos\theta \right\}$$



$$\iint_D \sqrt{x^2+y^2}\, dx\, dy$$

$$= \iint_E \rho\, \rho\, d\rho\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} \rho^2\, d\rho \right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3} \underbrace{\cos^3\theta}_{\text{pari!}}\, d\theta$$

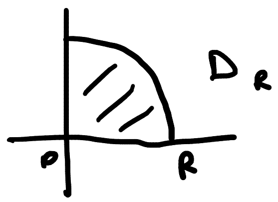
$$= \frac{16}{3} \int_0^{\pi/2} (1-\sin^2\theta)\cos\theta\, d\theta \underset{t=\sin\theta}{=} \cdots$$

Digressione:

$$\int_{-\infty}^{+\infty} \underbrace{e^{-t^2}}_{\text{pari}}\, dt = 2\int_0^{+\infty} e^{-t^2}\, dt = 2 \lim_{R\to+\infty} \int_0^R e^{-t^2}\, dt$$

non ho primitiva (esprimibile ...)

$\forall R > 0$:

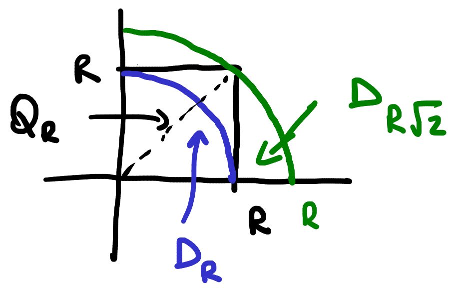


$$\iint_{D_R} e^{-(x^2+y^2)}\, dx\, dy \underset{\text{coord. polari}}{=} \iint_{[0,R]\times[0,\frac{\pi}{2}]} e^{-\rho^2}\, \rho\, d\rho$$

$$= \int_0^R e^{-\rho^2} \rho \, d\rho \cdot \int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{2} \left[-\frac{e^{-\rho^2}}{2} \right]_0^R$$

$$= \frac{\pi}{2} \, \frac{1 - e^{-R^2}}{2}$$

$$\Rightarrow \lim_{R \to +\infty} \iint_{D_R} e^{-(x^2+y^2)} \, dx \, dy = \frac{\pi}{4}$$

- 

$$D_R \subseteq Q_R \subseteq D_{R\sqrt{2}}$$

$\forall R > 0$:

$$\Rightarrow \underbrace{\iint_{D_R} e^{-(x^2+y^2)} \, dx \, dy}_{\downarrow \; \frac{\pi}{4}} \leq \iint_{Q_R} e^{-(x^2+y^2)} \, dx \, dy \leq \underbrace{\iint_{D_{R\sqrt{2}}} e^{-(x^2+y^2)} \, dx \, dy}_{\downarrow \; \frac{\pi}{4}}$$

$R \to +\infty$

TCO

$$\Rightarrow \lim_{R \to +\infty} \iint_{Q_R} e^{-(x^2+y^2)} \, dx \, dy = \frac{\pi}{4}$$

Oss:

$$\iint_{Q_R} e^{-(x^2+y^2)} \, dx \, dy = \iint_{[0,R] \times [0,R]} e^{-x^2} e^{-y^2} \, dx \, dy$$

$$= \int_0^R e^{-x^2} dx \cdot \int_0^R e^{-y^2} dy = \left(\int_0^R e^{-t^2} dt \right)^2$$

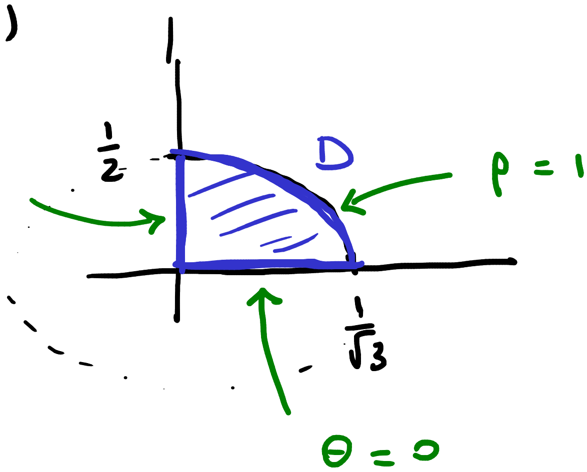
Dato che: $\left(\int_0^R e^{-t^2} dt \right)^2 \xrightarrow[R \to +\infty]{} \frac{\pi}{4}$, deduco $\displaystyle\int_0^R e^{-t^2}\,dt \longrightarrow \frac{\sqrt{\pi}}{2}$

Quindi:

$$\underbrace{\int_{-\infty}^{+\infty} e^{-t^2}\,dt}_{} = 2\lim_{R\to+\infty}\int_0^R e^{-t^2}\,dt = \underbrace{\sqrt{\pi}}_{}$$

Esempi (coord. ellittiche)

$f(x,y) = x+y$



$3x^2 + 4y^2 = 1$

$$\frac{x^2}{\left(\frac{1}{\sqrt{3}}\right)^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$$

$$\Phi(\rho,\theta) = \left(\frac{1}{\sqrt{3}}\rho\cos\theta,\ \frac{1}{2}\rho\sin\theta\right)$$

$$3x^2+4y^2=1 \iff \cancel{3}\cdot\frac{1}{\cancel{3}}\rho^2\cos^2\theta + \cancel{4}\cdot\frac{1}{\cancel{4}}\rho^2\sin^2\theta = 1$$

$$\iff \rho^2 = 1 \underset{(\rho\geq 0)}{\iff} \rho = 1$$

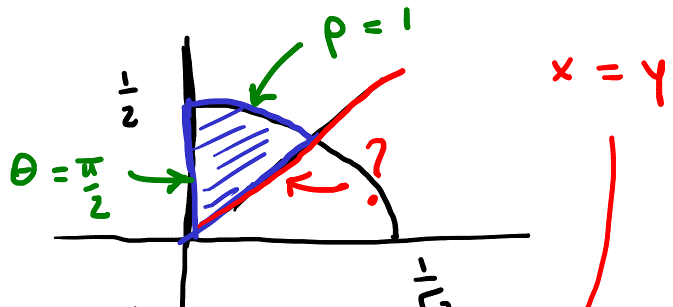
$$\Phi^{-1}(D) = \left\{(\rho,\theta) \;\middle|\; 0\leq\rho\leq 1,\ 0\leq\theta\leq\frac{\pi}{2}\right\}$$

$$\iint_D (x+y)\,dx\,dy = \iint_{[0,1]\times\left[0,\frac{\pi}{2}\right]} \left(\frac{1}{\sqrt{3}}\rho\cos\theta + \frac{1}{2}\rho\sin\theta\right)\frac{1}{\sqrt{3}}\cdot\frac{1}{2}\cdot\rho\,d\rho\,d\theta$$

$$= \frac{1}{2\sqrt{3}}\int_0^1 \rho^2\,d\rho \cdot \int_0^{\pi/2}\left(\frac{1}{\sqrt{3}}\cos\theta + \frac{1}{2}\sin\theta\right)d\theta = \ldots$$

- $f(x,y) = x + y$

Stesso cambio di variabili dell'es. precedente



$$\frac{1}{\sqrt{3}} \rho \cos\theta = \frac{1}{2} \rho \sin\theta$$

$$\frac{1}{\sqrt{3}} \cos\theta = \frac{1}{2} \sin\theta$$

$$\tan\theta = \frac{2}{\sqrt{3}}$$

$$\theta = \arctan \frac{2}{\sqrt{3}}$$

$$\Phi^{-1}(D) = \left\{ (\rho, \theta) \;\middle|\; 0 \le \rho \le 1 ,\ \arctan \frac{2}{\sqrt{3}} \le \theta \le \frac{\pi}{2} \right\}$$

$$\iint_D (x+y)\,dx\,dy = \dots = \frac{1}{2\sqrt{3}} \int_0^1 \rho^2 \, d\rho \cdot \int_{\arctan \frac{2}{\sqrt{3}}}^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{3}} \cos\theta + \frac{1}{2} \sin\theta \right) d\theta$$

$$= \dots$$