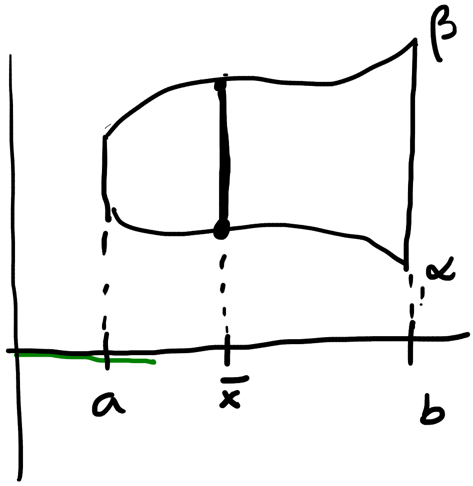
24 maggio 2024

Qualche commento sulle formule di riduzione

$f(\bar{x}, y)$, $y \in [\alpha(\bar{x}), \beta(\bar{x})]$

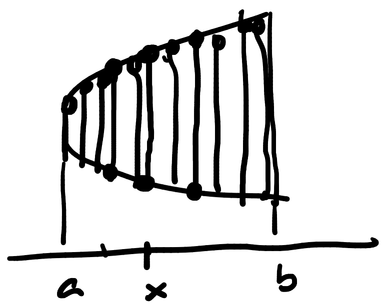


$$\underbrace{\int_{\alpha(\bar{x})}^{\beta(\bar{x})} f(\bar{x}, y)\, dy}_{\in \mathbb{R}, \text{ dipende da } \bar{x}}$$

$$\varphi := x \in [a,b] \longmapsto \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy \in \mathbb{R}$$

è continua $\Rightarrow$ integrabile in $[a,b]$

$\Rightarrow$ posso considerare $\int_a^b \varphi(x)\, dx \in \mathbb{R}$



Esempi (formule di riduzione per int. doppi)

- $D = \{(x,y) \mid a \le x \le b,\ \alpha(x) \le y \le \beta(x)\}$

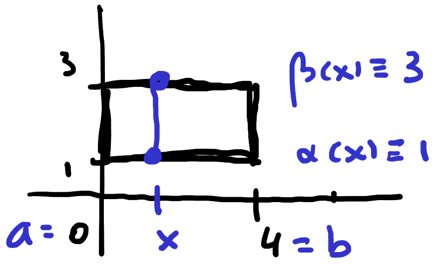
$$m_2(D) \overset{\text{oss.}}{=} \iint_D 1\, dx\, dy = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} 1\, dy \right) dx$$

$$= \int_a^b (\beta(x) - \alpha(x))\, dx$$

in accordo con la definizione data

- Integrale di: $f(x,y) = x^2 + xy$

  in $D = [0,4] \times [1,3]$



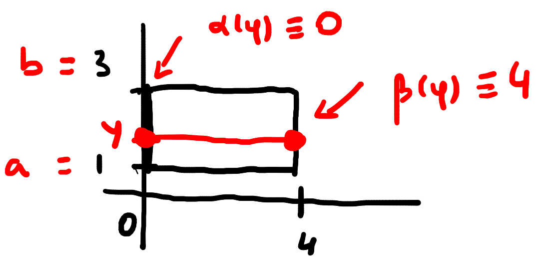
$$\iint_D f(x,y)\, dx\, dy = \int_0^4 \left(\int_1^3 (x^2 + xy)\, dy \right) dx$$

$$= \int_0^4 \left[x^2 y + x \frac{y^2}{2} \right]_1^3 dx$$

$$= \int_0^4 \left(3x^2 + \frac{9}{2} x - x^2 - \frac{x}{2} \right) dx = \int_0^4 (2x^2 + 4x)\, dx$$

$$= \left[\frac{2}{3} x^3 + 2x^2 \right]_0^4 = \frac{128}{3} + 32$$

In alternativa:



$$\iint_D f(x,y)\, dx\, dy = \int_1^3 \left(\int_0^4 (x^2 + xy)\, dx \right) dy$$

$$= \int_1^3 \left[\frac{x^3}{3} + \frac{x^2}{2} y \right]_0^4 dy = \int_1^3 \left(\frac{64}{3} + 8y \right) dy$$

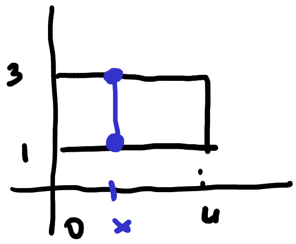
$$= \left[\frac{64}{3} y + 4y^2 \right]_1^3 = \frac{128}{3} + 32 \quad ✓$$

OSS: $f : [a,b] \times [c,d] \to \mathbb{R}$ continua

$$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

formula di inversione dell' ordine di integrazione

- $$\iint_{[0,4]\times[1,3]} x y^2 \, dx\, dy$$



$$= \int_0^4 \left(\int_1^3 \underbrace{x}_{\uparrow} y^2 \, dy \right) dx$$

linearità — non dipende da y

$$\overset{\downarrow}{=} \int_0^4 x \underbrace{\left(\int_1^3 y^2 dy\right)}_{\uparrow} dx \quad \overset{\downarrow}{=} \quad \int_0^4 x\,dx \cdot \int_1^3 y^2 dy$$

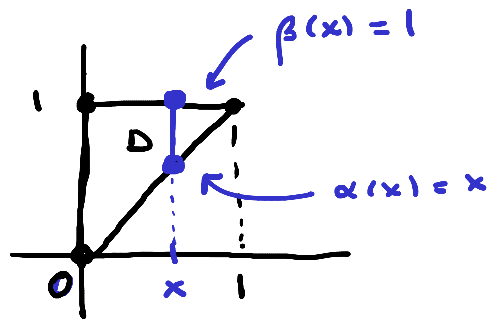
non dipende da x — linearità

$$= \left[\frac{x^2}{2}\right]_0^4 \cdot \left[\frac{y^3}{3}\right]_1^3 = \dots$$

Oss:

$$\iint_{[a,b]\times[c,d]} g(x)\, h(y)\, dx\, dy = \int_a^b g(x) dx \cdot \int_c^d h(y) dy$$

- $$\iint_D x y^2 \, dx\, dy$$

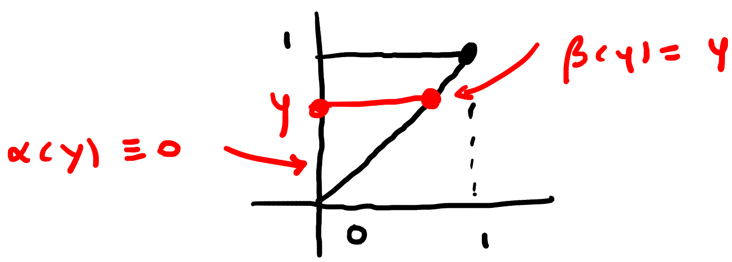


$$= \int_0^1 \left(\int_x^1 x y^2 dy \right) dx$$

$$= \int_0^1 \left[x \frac{y^3}{3} \right]_x^1 dx = \int_0^1 \left(\frac{x}{3} - \frac{x^4}{3} \right) dx = \left[\frac{x^2}{6} - \frac{x^5}{15} \right]_0^1$$

$$= \frac{1}{6} - \frac{1}{15} = \frac{1}{10}$$

In alternativa:



$$\iint_D f(x,y)\, dx\, dy =$$

$$\int_0^1 \left(\int_0^y x y^2 \, dx \right) dy = \int_0^1 \left[\frac{x^2}{2} y^2 \right]_0^y dy$$

$$= \int_0^1 \frac{y^4}{2} \, dy = \left[\frac{y^5}{10} \right]_0^1 = \frac{1}{10} \quad \checkmark$$

- D come prima, $f(x,y) = e^{y^2}$

D normale rispetto all' asse x:

$$\iint_D f(x,y) \, dx dy = \int_0^1 \Big(\underbrace{\int_x^1 e^{y^2} dy}_{?!!} \Big) dx$$

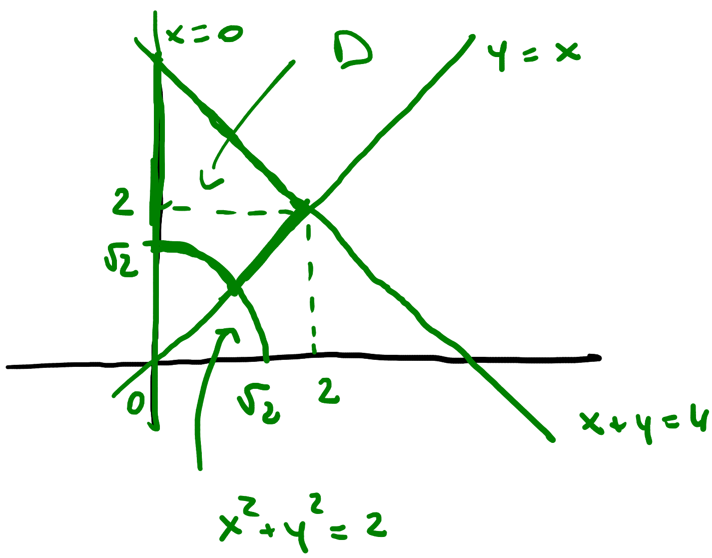
Provo con D normale rispetto all'asse y:

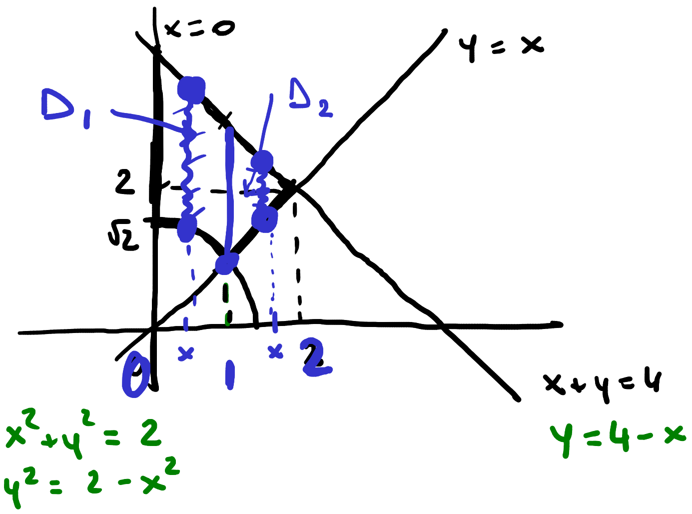
$$\iint_D f(x,y) \, dx dy = \int_0^1 \left(\int_0^y e^{y^2} dx \right) dy$$

$$= \int_0^1 e^{y^2} y \, dy = \frac{1}{2} \int_0^1 e^{y^2} (2y) \, dy$$

$$= \frac{1}{2} \left[e^{y^2} \right]_0^1 = \frac{e-1}{2} .$$

- $\iint_D \frac{x}{y} \, dx dy$





$$D = D_1 \cup D_2$$

$$D_1 = \{ (x,y) \mid 0 \le x \le 1 \,,\ \sqrt{2-x^2} \le y \le 4-x \}$$

$$D_2 = \{ (x,y) \mid 1 \le x \le 2 \,,\ x \le y \le 4-x \}$$

additività

$$\iint_D \frac{x}{y}\, dx\, dy = \underbrace{\iint_{D_1} \frac{x}{y}\, dx\, dy}_{=: I_1} + \underbrace{\iint_{D_2} \frac{x}{y}\, dx\, dy}_{=: I_2}$$

$$I_1 = \int_0^1 \left(\int_{\sqrt{2-x^2}}^{4-x} \frac{x}{y}\, dy \right) dx$$

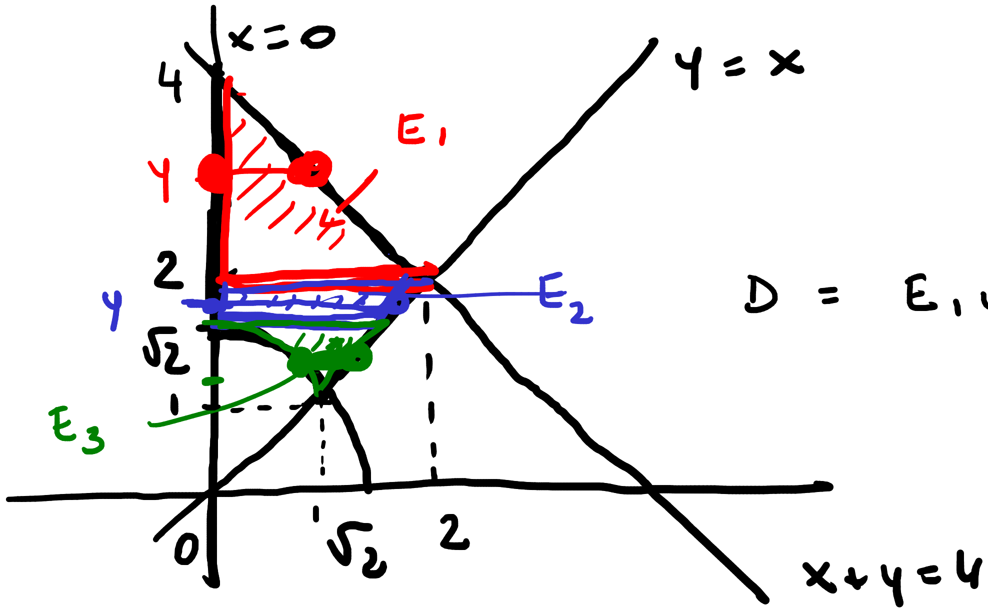
$$= \int_0^1 \left[x \ln(y) \right]_{\sqrt{2-x^2}}^{4-x} dx$$

$$= \int_0^1 \left(x \underbrace{\ln(4-x)}_{\text{per parti}} - \frac{x}{2} \underbrace{\ln(2-x^2)}_{\text{sostit + per parti}} \right) dx = \ldots$$

$$I_2 = \int_1^2 \left(\int_x^{4-x} \frac{x}{y}\, dy \right) dx = \int_1^2 \left(x \ln(4-x) - x \ln(x) \right) dx$$

$$= \ldots$$

In alternativa: decompongo D nell'unione di domini normali rispetto all'asse y (sperando che gli integrali da calcolare siano più semplici)



$$D = E_1 \cup E_2 \cup E_3$$

$$E_1 = \{(x,y) \mid 2 \le y \le 4, \quad 0 \le x \le 4-y\}$$

$$E_2 = \{(x,y) \mid \sqrt{2} \le y \le 2, \quad 0 \le x \le y\}$$

$$E_3 = \{(x,y) \mid 1 \le y \le \sqrt{2}, \quad \sqrt{2-y^2} \le x \le y\}$$

$$\iint_D \frac{x}{y}\,dx\,dy = \underbrace{\iint_{E_1} \frac{x}{y}\,dx\,dy}_{J_1} + \underbrace{\iint_{E_2} \frac{x}{y}\,dx\,dy}_{J_2} + \underbrace{\iint_{E_3} \frac{x}{y}\,dx\,dy}_{J_3}$$

$$J_1 = \int_2^4 \left(\int_0^{4-y} \frac{x}{y}\,dx \right) dy = \int_2^4 \left[\frac{x^2}{2y} \right]_0^{4-y} dy$$

$$= \int_2^4 \frac{(4-y)^2}{2y}\,dy = \int_2^4 \left(\frac{8}{y} - 4 + \frac{y}{2} \right) dy = \ldots \text{ facile!}$$

$$J_2 = \int_{\sqrt{2}}^{2} \left(\int_0^y \frac{x}{y}\,dx \right) dy = \int_{\sqrt{2}}^{2} \frac{y}{2}\,dy = \ldots \checkmark$$

$$J_3 = \int_1^{\sqrt{2}} \left(\int_{\sqrt{2-y^2}}^{y} \frac{x}{y}\, dx \right) dy = \int_1^{\sqrt{2}} \left[\frac{x^2}{2y} \right]_{\sqrt{2-y^2}}^{y} dy$$

$$= \int_1^{\sqrt{2}} \left(\frac{y}{2} - \frac{2-y^2}{2y} \right) dy = \int_1^{\sqrt{2}} \left(\frac{y}{2} - \frac{1}{y} + \frac{y}{2} \right) dy = \ldots \checkmark$$

$\square$

Esempi (formule di integrazione per fili)

- $T = \{ (x,y,z) \mid (x,y) \in D,\ \gamma(x,y) \le z \le \delta(x,y) \}$

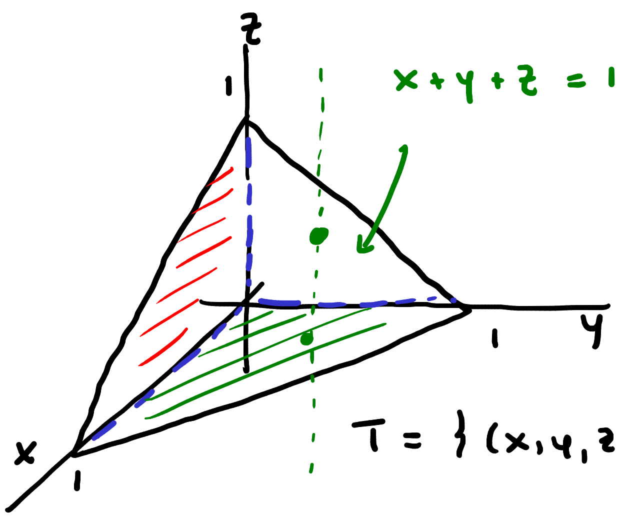
$$m_3(T) \overset{\text{oss.}}{=} \iiint_T 1 \, dx\, dy\, dz = \iint_D \left(\int_{\gamma(x,y)}^{\delta(x,y)} 1 \, dz \right) dx\, dy$$

$$= \iint_D \left(\delta(x,y) - \gamma(x,y) \right) dx\, dy$$

coerente con la definizione data

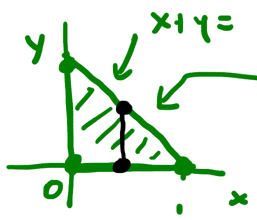
- Integrale di $f(x,y,z) = x + z$

T : tetraedro di vertici $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$



Descrivo T come insieme normale rispetto al piano xy:

$$T = \{ (x,y,z) \mid (x,y) \in \underline{D},\ 0 \le z \le 1-x-y \}$$



con $\underline{D} = \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x \}$

$$\iiint_T (x+z)\,dx\,dy\,dz = \iint_D \left(\int_0^{1-x-y} (x+z)\,dz \right) dx\,dy$$

$$= \iint_D \left[xz + \frac{z^2}{2} \right]_0^{1-x-y} dx\,dy$$

$$= \iint_D \left(x(1-x-y) + \frac{(1-x-y)^2}{2} \right) dx\,dy$$

$$= \iint_D \left(\underbrace{x(1-x)} - xy + \underbrace{\frac{(1-x)^2}{2}} - \frac{2(1-x)y}{2} + \frac{y^2}{2} \right) dx\,dy$$

$$= \iint_D \left((1-x)\left(x + \frac{1-x}{2}\right) - \cancel{xy} - (1-\cancel{x})y + \frac{y^2}{2} \right) dx\,dy$$

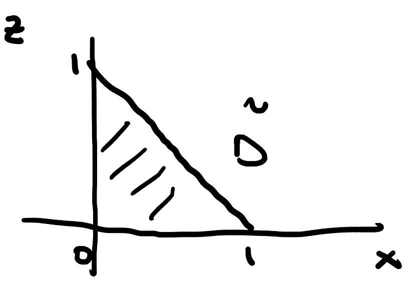
$$= \int_0^1 \left(\int_0^{1-x} \left((1-x)\frac{1+x}{2} - y + \frac{y^2}{2} \right) dy \right) dx$$

$$= \int_0^1 \left(\frac{(1-x)^2(\cancel{1}+x)}{2} - \cancel{\frac{(1-x)^2}{2}} + \frac{(1-x)^3}{6} \right) dx = \ldots = \frac{1}{12}$$

(I conti fatti in aula erano corretti! ☺)

In alternativa: osservo che f non dipende esplicitamente da y e descrivo T come insieme normale rispetto al piano xz:

$$T = \{ (x,y,z) \mid (x,z) \in \tilde{D},\quad 0 \le y \le 1-x-z \}$$



$$\iiint_T (x+z)\,dx\,dy\,dz = \iint_{\tilde D} \left(\int_0^{1-x-z} (x+z)\,dy \right) dx\,dz$$

$$= \iint_{\tilde D} (x+z)(1-x-z)\,dx\,dz$$

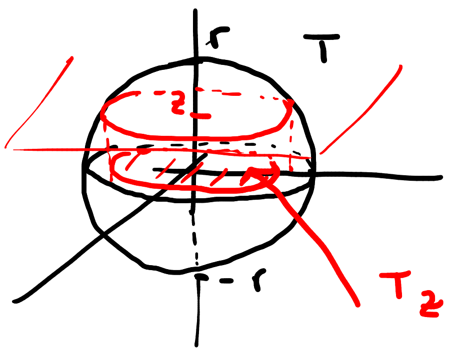
$$= \int_0^1 \left(\int_0^{1-x} \left(x(1-x) - xz + z - zx - z^2 \right) dz \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} \left(x(1-x) + z(1-2x) - z^2 \right) dz \right) dx$$

$$= \int_0^1 \left(\cancel{x(1-x)^2} + \frac{(1-\cancel{2x})(1-x)^2}{2} - \frac{(1-x)^3}{3} \right) dx$$

$$= \int_0^1 \left(\frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right) dx = \left[-\frac{(1-x)^3}{6} + \frac{(1-x)^4}{12} \right]_0^1 = \frac{1}{12}$$

Esempi (integrazione per strati)

- 

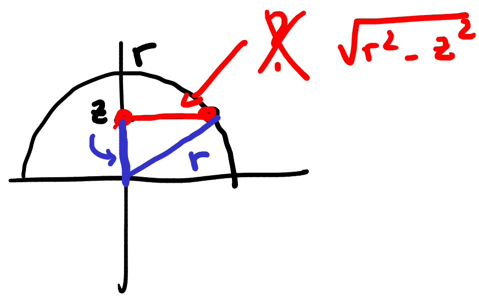
$\mathrm{vol}(T) = ?$

$\left(\overset{\shortparallel}{m_3(T)} \right)$

$$m_3(T) = \iiint_T 1\,dx\,dy\,dz = \int_{-r}^{r} \left(\iint_{T_z} 1\,dx\,dy \right) dz = (*)$$

$T_z = \cancel{?}$

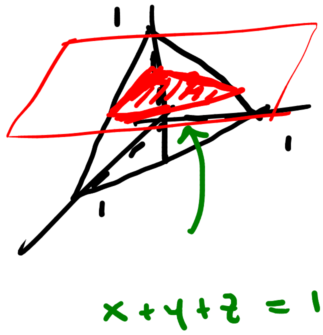
disco di centro $(0,0)$ e raggio $\sqrt{r^2 - z^2}$



$$(*) = \int_{-r}^{r} \underbrace{m_2(T_z)}_{\pi(r^2-z^2)}\,dz = \int_{-r}^{r} \pi(r^2 - z^2)\,dz$$

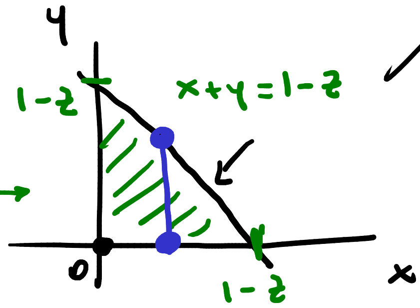
$$= 2\pi \int_0^r (r^2 - z^2)\,dz = 2\pi\left(r^2\cdot r - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3 \;\checkmark$$

- Ricalcolo $\iiint_T (x+z)\,dx\,dy\,dz$



$a = 0, \quad b = 1$

$\forall z \in [0,1]$: $\quad T_z = \cancel{?}$



$$\iiint_T (x+z)\,dx\,dy\,dz = \int_0^1 \underbrace{\left(\iint_{T_z} (x+z)\,dx\,dy\right)}_{?} dz$$

$$\iint_{T_z} (x+z)\,dx\,dy = \int_0^{1-z}\left(\int_0^{1-z-x} (x+z)\,dy\right)dx$$

$$= \int_0^{1-z} (x+z)(1-z-x)\,dx$$

$$= \int_0^{1-z} \left(z(1-z) + x - xz - xz - x^2\right)dx$$

$$= \int_0^{1-z} \left(z(1-z) + (1-2z)x - x^2\right)dx$$

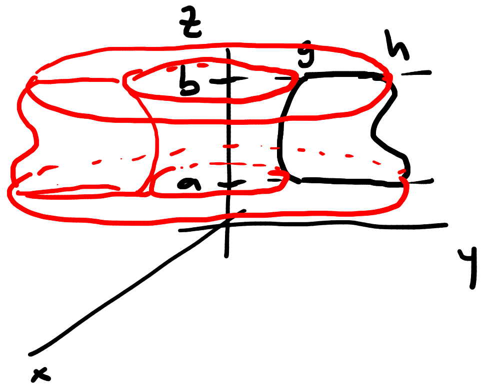
$$= z(1-z)^2 + (1-2z)\frac{(1-z)^2}{2} - \frac{(1-z)^3}{3}$$

$$= \left(z + \frac{1}{2} - z\right)(1-z)^2 - \frac{(1-z)^3}{3}$$

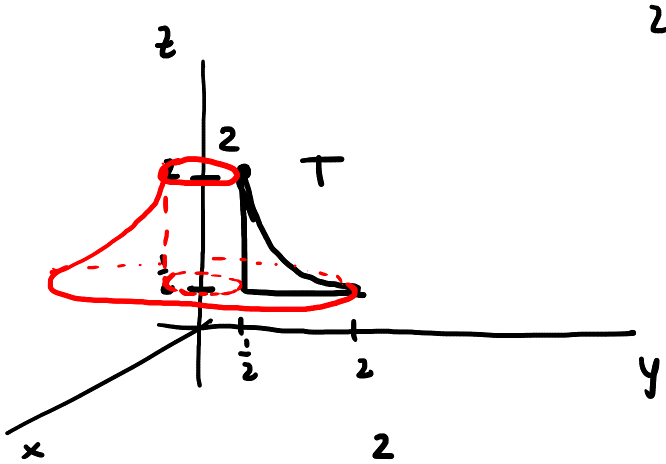
$$= \frac{(1-z)^2}{2} - \frac{(1-z)^3}{3}$$

Int. triplo $= \int_0^1 \left(\frac{(1-z)^2}{2} - \frac{(1-z)^3}{3}\right)dz = \ldots$

Volume dei solidi di rotazione



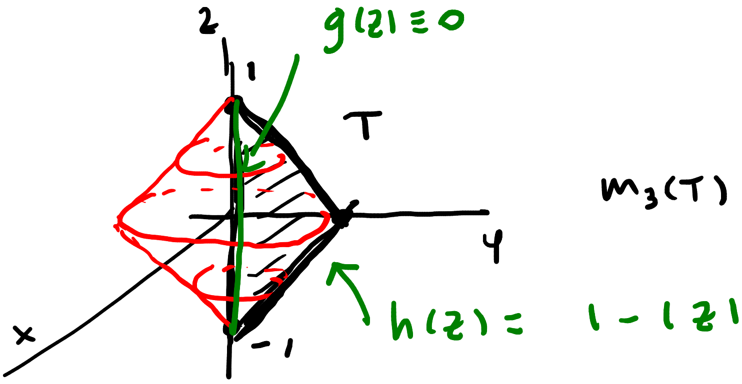
$$\{(0, y, z) \mid a \le z \le b,\ g(z) \le y \le h(z)\}$$



$$\tfrac{1}{2} \le z \le 2, \qquad \tfrac{1}{2} \le y \le \tfrac{1}{z}$$

$$m_3(T) = \pi \int_{\frac{1}{2}}^{2} \left(\frac{1}{z^2} - \frac{1}{4} \right) dz = \pi \left[-\frac{1}{z} - \frac{z}{4} \right]_{\frac{1}{2}}^{2} = \dots$$

$$= \frac{9}{8}\pi$$



$$m_3(T) = \pi \int_{-1}^{1} \left((1-|z|)^2 - 0^2 \right) dz$$

$$= 2\pi \int_0^1 (1-z)^2 \, dz$$

$$= 2\pi \left[-\frac{(1-z)^3}{3} \right]_0^1$$

$$= \frac{2}{3}\pi \qquad \square$$