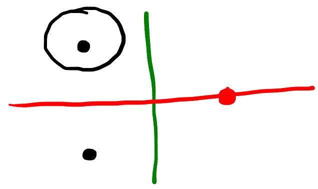
7 maggio 2024

Es. (rilevanza dell' ordine di derivazione)

$$f(x,y) = \begin{cases} y^2 \arctan\left(\frac{x}{y}\right) & (x,y) \in \mathbb{R} \times \mathbb{R}^* \\ 0 & (x,y) \in \mathbb{R} \times \{0\} \end{cases}$$



$\forall (x,y) \in \mathbb{R}^2$:

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} y \neq 0 & y^2 \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y^3}{x^2+y^2} \\ y = 0 & \lim_{t \to 0} \frac{\overset{=0}{f(x+t,0)} - \overset{=0}{f(x,0)}}{t} = 0 \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) \cancel{\overset{?}{=}} \lim_{t \to 0} \frac{\frac{\partial f}{\partial x}(0,t) - \frac{\partial f}{\partial x}(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\frac{t^3}{0^2+t^2} - 0}{t} = \lim_{t \to 0} \frac{t}{t} = 1$$

$\forall (x,y) \in \mathbb{R}^2$:

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} y \neq 0 & 2y \arctan\left(\frac{x}{y}\right) + y^2 \frac{1}{1+\frac{x^2}{y^2}} \left(-\frac{x}{y^2}\right) = \dots \\ y = 0 & \lim_{t \to 0} \frac{f(x,t) - f(x,0)}{t} = \end{cases}$$

$$\lim_{t \to 0} \frac{\underset{\downarrow 0}{t^2} \overbrace{\arctan\left(\frac{x}{t}\right)}^{\text{limitata}} - 0}{\cancel{t}} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \overset{\cancel{?}}{=} \lim_{t \to 0} \frac{\overset{=0}{\frac{\partial f}{\partial y}(t,0)} - \overset{=0}{\frac{\partial f}{\partial y}(0,0)}}{t} = 0$$

Es. $f(x,y,z) = x^5 + y^4 z^3 - 3x z^2 \qquad (x,y,z) \in \mathbb{R}^3$

Oss: f polinomiale $\Rightarrow$ f di classe C^2

$\forall (x,y,z) \in \mathbb{R}^3$:

$$\frac{\partial f}{\partial x}(x,y,z) = 5x^4 - 3z^2$$

$$\frac{\partial f}{\partial y}(x,y,z) = 4y^3 z^3$$

$$\frac{\partial f}{\partial z}(x,y,z) = 3y^4 z^2 - 6xz$$

$$H_f(x,y,z) = \begin{pmatrix} 20x^3 & 0 & -6z \\ 0 & 12y^2z^3 & 12y^3z^2 \\ -6z & 12y^3z^2 & 6y^4z - 6x \end{pmatrix}$$

Motivazione per polinomio di Taylor

Suppongo f di classe C^2 in $B_r(\bar{x})$, $r > 0$.

Fisso $x \in B_r(\bar{x})$, $x \neq \bar{x}$

Definisco

$$g(t) = f(\bar{x} + t(x - \bar{x}))$$

con $t \in \left(-\frac{r}{\|x-\bar{x}\|}, \underset{>1}{\frac{r}{\|x-\bar{x}\|}}\right)$ $\quad (\Rightarrow \bar{x} + t(x-\bar{x}) \in B_r(\bar{x}))$

Osservo che $[0,1] \subset \left(-\frac{r}{\|x-\bar{x}\|}, \frac{r}{\|x-\bar{x}\|}\right)$

Inoltre: g è composta di funzioni di classe C^2 e quindi è di classe C^2.

Scrivo il pol. di Taylor di centro 0 e ordine 2 di g:

$$T_{0,2}(t) = g(0) + g'(0)\,t + \frac{1}{2} g''(0)\,t^2$$

Per $t = 1$:

$$T_{0,2}(1) = g(0) + g'(0) + \frac{1}{2} g''(0)$$

$$g(t) := f(\bar{x} + t(x-\bar{x})) \qquad g(0) = f(\bar{x})$$

$$g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{x} + t(x-\bar{x}))\cdot \overbrace{(x_i - \bar{x}_i)}^{\text{non dipende da } t} \qquad g'(0) = \nabla f(\bar{x})\cdot(x-\bar{x})$$

$$g''(t) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_i}(\bar{x} + t(x-\bar{x}))(x_j - \bar{x}_j) \right)(x_i - \bar{x}_i)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(\bar{x} + t(x-\bar{x}))(x_j - \bar{x}_j)(x_i - \bar{x}_i)$$

$$\Rightarrow g''(0) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{x})(x_i - \bar{x}_i)(x_j - \bar{x}_j)$$

$$= H_f(\bar{x})(x-\bar{x})\cdot(x-\bar{x})$$

Sostituendo:

$$T_{0,2}(1) = f(\bar{x}) + \nabla f(\bar{x})\cdot(x-\bar{x}) + \frac{1}{2} H_f(\bar{x})(x-\bar{x})\cdot(x-\bar{x})$$

$\square$

Es: $f(x,y) = \dfrac{\cos x}{\cos y}$ $\quad (x,y) \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

?? pol. di Taylor di centro $(0,0)$ e ordine 2

- $f(0,0) = 1$

- $\dfrac{\partial f}{\partial x}(x,y) = -\dfrac{\sin x}{\cos y}$ $\qquad \dfrac{\partial f}{\partial x}(0,0) = 0$

$\dfrac{\partial f}{\partial y}(x,y) = \cos x \left(-\dfrac{-\sin y}{\cos^2 y}\right) = \cos x \dfrac{\sin y}{\cos^2 y}$

$\dfrac{\partial f}{\partial y}(0,0) = 0$

$\Rightarrow \quad \nabla f(0,0) = (0,0)$

- $\dfrac{\partial^2 f}{\partial x^2}(x,y) = -\dfrac{\cos x}{\cos y}$ $\quad \Rightarrow \quad \dfrac{\partial^2 f}{\partial x^2}(0,0) = -1$

$\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y) = -\sin x \dfrac{\sin y}{\cos^2 y}$

$\Rightarrow \dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 0$

$\dfrac{\partial^2 f}{\partial y^2}(x,y) = \cos x \cdot \dfrac{\cos y \cdot \cos^2 y - \sin y \, 2\cos y(-\sin y)}{\cos^4 y}$

$\Rightarrow \dfrac{\partial^2 f}{\partial y^2}(0,0) = 1$

$T_{(0,0),2}(x,y) = f(0,0) + \nabla f(0,0) \cdot (x,y) +$

$+ \frac{1}{2} H_f(0,0) \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$$= 1 + 0 + \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + \frac{1}{2}\begin{pmatrix} -x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 1 + \frac{1}{2}\left(-x^2 + y^2\right)$$

Formula di Taylor:

$$\frac{\cos x}{\cos y} = 1 - \frac{1}{2}x^2 + \frac{1}{2}y^2 + o\left(x^2+y^2\right)$$

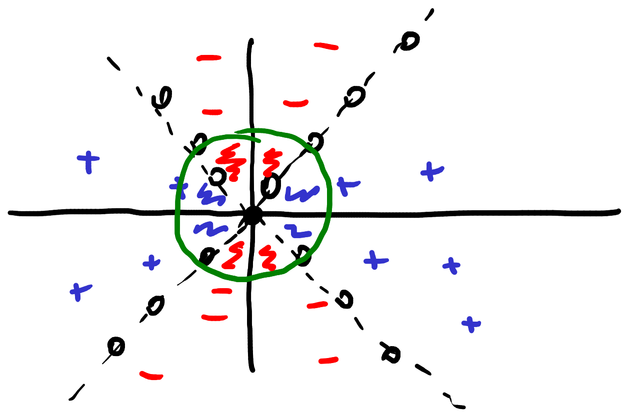
Esempi (di estremi)

- $f(x,y) = x^2+y^2$

$$\left.\begin{array}{l} f(0,0) = 0 \\ f(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2 \end{array}\right| \Rightarrow \quad (0,0) \text{ punto di min. } \underline{\text{globale}}$$

- $f(x,y) = \sqrt{x^2+y^2}$ idem!

- $f(x,y) = x^2 - y^2$



$f(0,0) = 0$

? $f(x,y) \geq f(0,0) \iff$

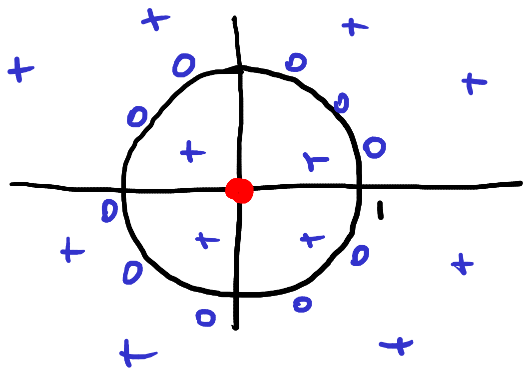
$x^2 - y^2 \geq 0 \iff$

$x^2 \geq y^2 \iff |y| \leq |x|$

OSS: $\forall\, U$ intorno di $(0,0)$ contiene punti in cui f vale di più che in $(0,0)$ e punti in cui f vale di meno che in $(0,0)$

$\Rightarrow$ $(0,0)$ non è punto di estremo

- $f(x,y) = (x^2+y^2-1)^2$



Tutti i punti della circonferenza unitaria sono di minimo globale

OSS: $f(x,y) \to +\infty$ se $\|(x,y)\| \to +\infty$

$\Rightarrow$ f non ha max globale

OSS: $\left.\begin{array}{l} f(0,0) = 1 \\ \forall (x,y) \in B_1(0,0) : f(x,y) \le 1 \end{array}\right\} \Rightarrow$

$(0,0)$ punto di max locale.

$f(x,y) = x^2 - y^2$

$\frac{\partial f}{\partial x}(x,y) = 2x$ $\qquad$ $\frac{\partial f}{\partial y}(x,y) = -2y$