3 maggio 2024

Considero la superficie cilindrica

Fisso $r, h \in \mathbb{R}_+^*$.

$K := [0, 2\pi] \times [0, h]$ ← dominio regolare ✓

$\sigma : K \to \mathbb{R}^3$ t.c. $\sigma(\theta, z) = (r\cos\theta, r\sin\theta, z)$

$\forall z \in [0, h]$: $\sigma(0, z) = \sigma(2\pi, z)$

$\Rightarrow \sigma$ non è iniettiva in K !!

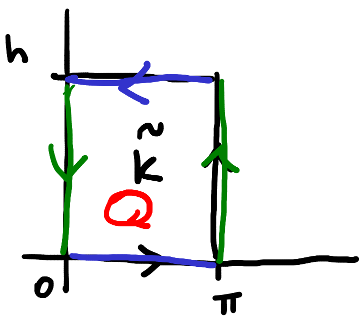
Pongo $\tilde{K} := [0, \pi] \times [0, h]$ e restringo

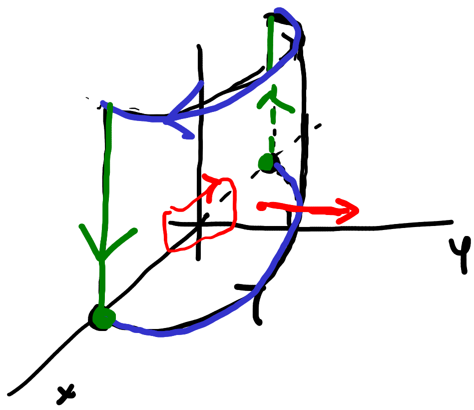
σ a $\tilde{K}$; osservo che $\sigma_{|\tilde{K}}$ è iniettiva

Già noto: $\forall (\theta, z) \in \mathbb{R} \times \mathbb{R}$:

$$N_\sigma(\theta, z) = (r\cos\theta, r\sin\theta, 0) \neq (0,0,0)$$

$\Rightarrow \sigma_{|\tilde{K}}$ soddisfa la seconda condizione nella def. di sup. regolare con bordo.





$r(t)$ $\qquad$ $\varphi(t) := \sigma(r(t))$

$0 \le t \le 1$ • $(\pi t, 0)$ $\qquad$ $(r\cos(\pi t), r\sin(\pi t), 0)$

$1 < t \le 2$ • $(\pi, h(t-1))$ $\qquad$ $(r\cos\pi, r\sin\pi, h(t-1))$

$(-r, 0, h(t-1))$

$2 < t \le 3$ • $(\pi(3-t), h)$ $\quad (r\cos(\pi(3-t)), r\sin(\pi(3-t)), h)$

$3 < t \le 4$ • $(0, h(4-t))$ $\quad (r\cos 0, r\sin 0, h(4-t)) = (r, 0, h(4-t))$

Superficie grafico

$K \subseteq \mathbb{R}^2$ insieme di parametri

$f: K \to \mathbb{R}$ continua

$\sigma: K \to \mathbb{R}^3$ t.c. $\sigma(u,v) = \overbrace{(u, v, f(u,v))}^{\text{iniettiva ovunque}}$

"buona" $\Longleftarrow$ "buona"

Suppongo f di classe C^1 in K (con K aperto oppure K dominio).

Per ogni $(u,v) \in K$:

$$\frac{\partial \sigma}{\partial u}(u,v) = \left(1, 0, \frac{\partial f}{\partial u}(u,v)\right)$$

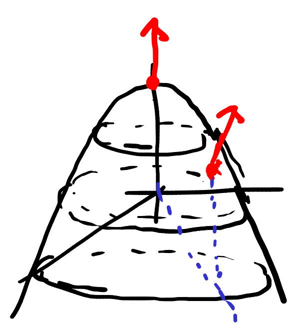
$$\frac{\partial \sigma}{\partial v}(u,v) = \left(0, 1, \frac{\partial f}{\partial v}(u,v)\right) \Rightarrow$$

$$N_\sigma(u,v) = \left(-\frac{\partial f}{\partial u}(u,v), -\frac{\partial f}{\partial v}(u,v), \underbrace{1}_{\neq 0}\right) \neq (0,0,0)$$

Es.

$f: \underbrace{\mathbb{R}^2}_{\text{aperto}} \to \mathbb{R}$ t.c. $f(x,y) = 1 - x^2 - y^2$

$\uparrow \in C^1$



$\Rightarrow$ la sup. grafico è regolare e orientabile

Parametrizzazione:

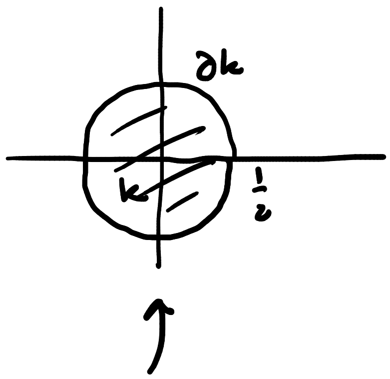
$\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ tc $\sigma(u,v) = (u, v, 1-u^2-v^2)$

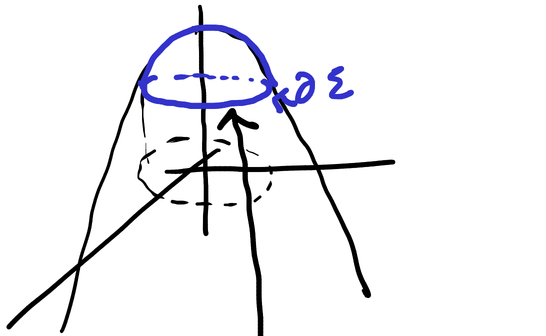
$\forall (u,v) \in \mathbb{R}^2$: $N_\sigma(u,v) = (2u, 2v, 1)$

- Stessa funzione, con $K = \bar{B}_{\frac{1}{2}}(0,0)$

$\uparrow$ dominio regolare

$\Rightarrow$ la sup. grafico è regolare con bordo





$t \in [0, 2\pi] \mapsto \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t\right)$

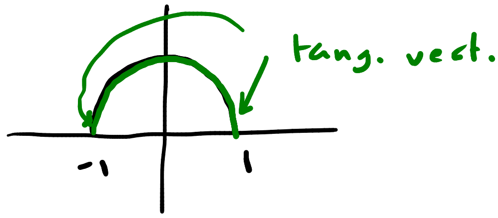
$t \in [0, 2\pi] \mapsto \sigma\left(\frac{1}{2}\cos t, \frac{1}{2}\sin t\right) =$

$= \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t, 1 - \frac{1}{4}\cos^2 t - \frac{1}{4}\sin^2 t\right)$

$= \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{3}{4}\right)$

- $f(x,y) = \sqrt{1-x^2-y^2}$

$\uparrow$ definita in $\bar{B}_1(0,0)$

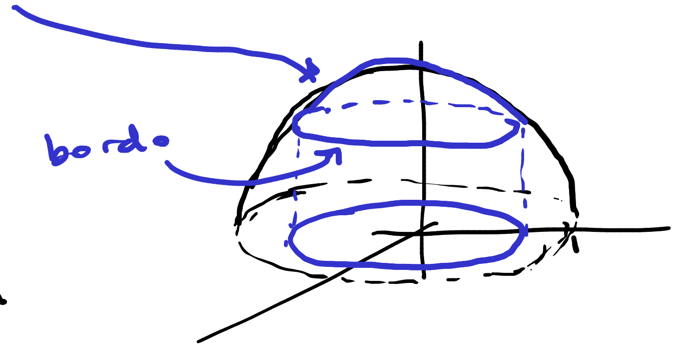


f non è di classe C^1 in $\bar{B}_1(0,0)$! (lo è in $B_1(0,0)$)

Fisso $r \in (0,1)$ e restringo f a $\overline{B}_r(0,0)$

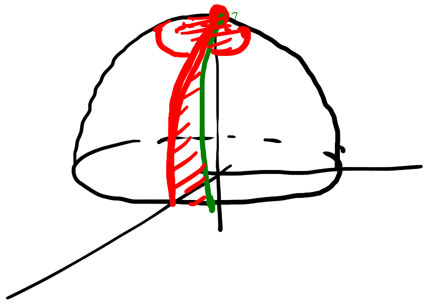
Osservo che $f|_{\overline{B}_r(0,0)}$ è di classe C^1

$\Rightarrow$ la sup. grafico associata è regolare con bordo.



Per $r \to 1^-$:

ottengo la semisfera superiore; la chiamo "regolare con bordo".

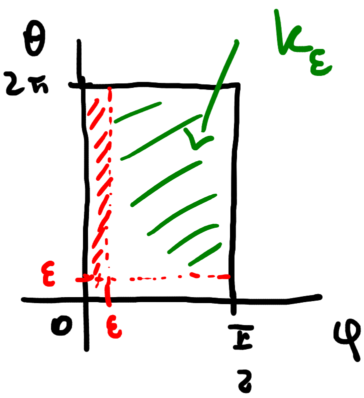


In alternativa:

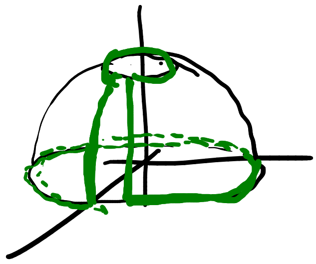
$$\sigma : \underbrace{\left[0, \tfrac{\pi}{2}\right]}_{\text{dom. reg. ✓}} \times [0, 2\pi] \to \mathbb{R}^3$$

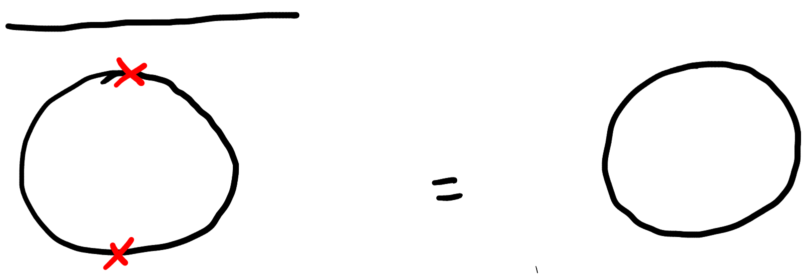
$$\sigma(\varphi, \theta) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

C^1 in $\mathbb{R}^2$ ✓



$\sigma|_{K_\varepsilon}$ è iniettiva





$r: [0,1] \subseteq \underbrace{\mathbb{R}} \to \mathbb{R}^n$

$f: A \subseteq \mathbb{R}^n \to \underbrace{\mathbb{R}}$

$g := f \circ r$

$$g'(t) = \left(\frac{\partial f}{\partial x_1}(r(t)), \dots, \frac{\partial f}{\partial x_n}(r(t)) \right) \begin{pmatrix} r_1'(t) \\ \vdots \\ r_n'(t) \end{pmatrix}$$

$$= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(r(t))\, r_j'(t)$$

Oss: se il teor. del valor medio valesse per $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ con $m \geq 2$, l'enunciato sarebbe:

$\exists\, z \in [x,y] \setminus \{x,y\}$ t.c.

$$f(y) - f(x) = J_f(z)\,(y-x)$$

Però: NON vale!

Es: $f: \mathbb{R} \to \mathbb{R}^2$ t.c. $f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

Se valesse il teorema, presi $0, 2\pi \in \mathbb{R}$ dovrebbe esistere $\tilde{t} \in (0, 2\pi)$ t.c.

$$f(2\pi) - f(0) = J_f(\tilde{t})\,(2\pi - 0)$$

cioè:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} -\sin \tilde{t} \\ \cos \tilde{t} \end{pmatrix} 2\pi$$

$\neq$ assurdo!