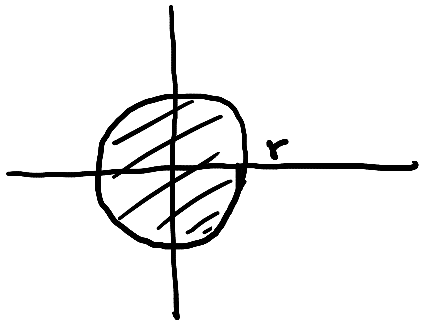
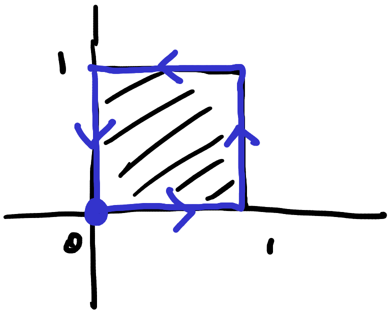
2 maggio 2024

Esempi di domini regolari in $\mathbb{R}^2$



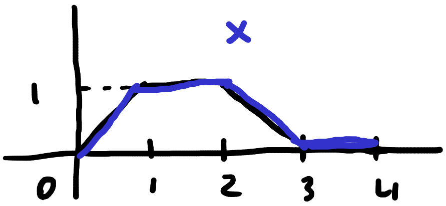
$D = \overline{B}_r(0,0)$

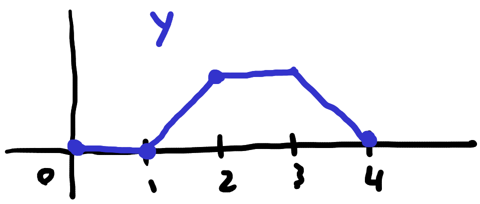
$\partial D: \quad r(t) = (r\cos t, r\sin t)$ $\quad t \in [0, 2\pi]$ regolare

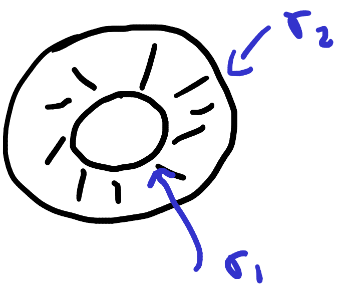


$$\partial D: \quad r(t) = \begin{cases} (t, 0) & t \in [0,1] \\ (1, t-1) & t \in (1,2] \\ (3-t, 1) & t \in (2,3] \\ (0, 4-t) & t \in (3,4] \end{cases}$$

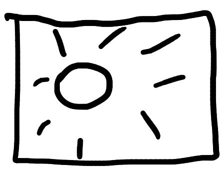
regolare a tratti





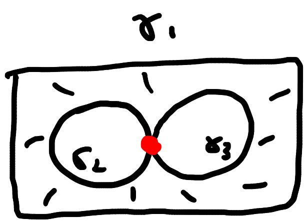


$\partial D = \gamma_1 \cup \gamma_2$ ✓



ok

ok



$\gamma_2 \cap \gamma_3 \neq \emptyset$

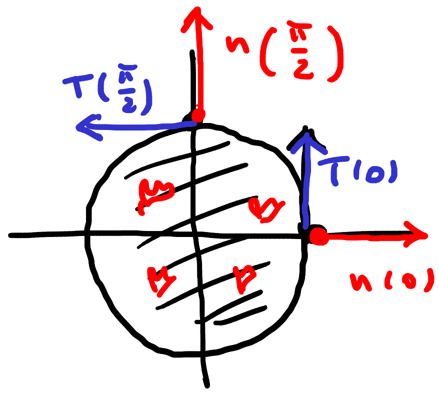
Es. sul versore normale

$r(t) = (\cos t, \sin t) \quad t \in [0, 2\pi]$

$r'(t) = (-\sin t, \cos t), \quad \|r'(t)\| = 1$

$\Rightarrow \quad T(t) = (-\sin t, \cos t)$

$n(t) = (\cos t, \sin t)$



Es: $r(0) = (1,0)$

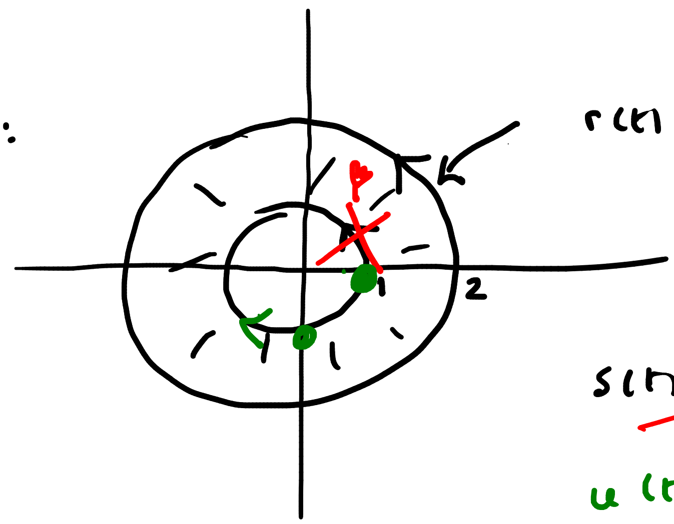
$T(0) = (0,1)$

$n(0) = (1,0)$

$r(\frac{\pi}{2}) = (0,1) \qquad T(\frac{\pi}{2}) = (-1,0)$

$n(\frac{\pi}{2}) = (0,1)$

Es:



$r(t) = (2\cos t, \ 2\sin t), \ t \in [0, 2\pi]$

$s(t) = \cancel{(\cos t, \sin t)} \quad t \in [0, 2\pi]$

$u(t) = (\cos(2\pi - t), \ \sin(2\pi - t))$

$t \in [0, 2\pi]$

$\sigma : K \subseteq \mathbb{R}^2 \to \mathbb{R}^3$

$\forall (u,v) \in \operatorname{int}(K)$:

$$\begin{pmatrix} \frac{\partial \sigma_1}{\partial u}(u,v) & \frac{\partial \sigma_1}{\partial v}(u,v) \\ \frac{\partial \sigma_2}{\partial u}(u,v) & \frac{\partial \sigma_2}{\partial v}(u,v) \\ \frac{\partial \sigma_3}{\partial u}(u,v) & \frac{\partial \sigma_3}{\partial v}(u,v) \end{pmatrix} = J_\sigma(u,v)$$

$a = (a_1, a_2, a_3) \qquad b = (b_1, b_2, b_3)$

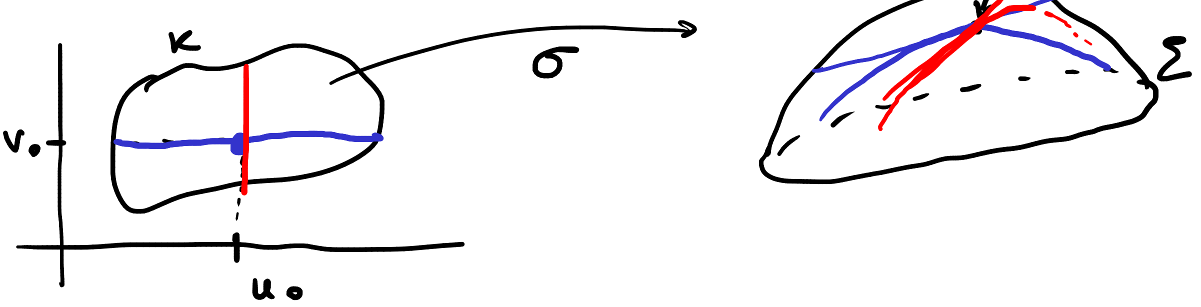
$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \text{ rango } 2 \iff$$

$$(a_2 b_3 - a_3 b_2,\ a_1 b_3 - a_3 b_1,\ a_1 b_2 - a_2 b_1) \neq (0,0,0)$$

$\iff a \times b \neq 0$

$\iff a, b$ lin. indipendenti

Perché piano "tangente"? $\sigma(u_0, v_0) =: P_0$



Derivo $u \mapsto \sigma(u, v_0)$ e ottengo $\dfrac{\partial \sigma}{\partial u}(u_0, v_0)$

Derivo $v \mapsto \sigma(u_0, v)$ e ottengo $\dfrac{\partial \sigma}{\partial v}(u_0, v_0)$

Fisso $r \in \mathbb{R}_+^*$

Verifico che la "superficie cilindrica" è regolare

$$\sigma : \underbrace{[0, 2\pi] \times \mathbb{R}}_{=: K} \to \mathbb{R}^3 \quad \text{t.c.}$$

$$\sigma(\theta, z) = (r\cos\theta,\ r\sin\theta,\ z)$$

tutte e tre di classe C^1 in $\mathbb{R}\times\mathbb{R}$

e quindi in $\mathrm{int}(K)$ ✓

Per ogni $(\theta, z) \in (0, 2\pi) \times \mathbb{R}$:

$$\begin{array}{lccc} & e_1 & e_2 & e_3 \\ \dfrac{\partial \sigma}{\partial \theta}(\theta, z) = & (-r\sin\theta, & r\cos\theta, & 0) \\ \dfrac{\partial \sigma}{\partial z}(\theta, z) = & (\ 0, & 0, & 1) \end{array}$$

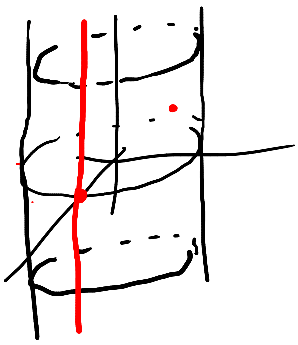
$$\Rightarrow \quad N_\sigma(\theta, z) = (r\cos\theta,\ r\sin\theta,\ 0)$$

$$\Rightarrow \quad \| N_\sigma(\theta, z) \| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 0^2} = r \neq 0$$

$\Rightarrow$ la superficie è regolare.

Per ogni $(\theta, z) \in (0, 2\pi) \times \mathbb{R}$:

$$n_\sigma(\theta, z) = \frac{N_\sigma(\theta, z)}{\| N_\sigma(\theta, z) \|} = \frac{(r\cos\theta, r\sin\theta, 0)}{r}$$



$$= \underbrace{(\cos\theta, \sin\theta, 0)}$$

ha senso (ed è continua) in $\mathbb{R}^2$, quindi in K

Verifico che è regolare la superficie sferica

$\sigma : \underbrace{[0,\pi] \times [0,2\pi]}_{K} \to \mathbb{R}^3$ t.c.

$\sigma(\varphi,\theta) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$

sono di classe C^1 in $\mathbb{R}^2$

(e quindi in int(K))

Per ogni $(\varphi,\theta) \in \text{int}(K) = (0,\pi) \times (0,2\pi)$:

$$\frac{\partial\sigma}{\partial\varphi}(\varphi,\theta) = (r\cos\varphi\cos\theta,\ r\cos\varphi\sin\theta,\ -r\sin\varphi)$$

$$\frac{\partial\sigma}{\partial\theta}(\varphi,\theta) = (-r\sin\varphi\sin\theta,\ r\sin\varphi\cos\theta,\ 0)$$

$$\Rightarrow N_\sigma(\varphi,\theta) = (r^2\sin^2\varphi\cos\theta,\ r^2\sin^2\varphi\sin\theta,\ r^2\cos\varphi\sin\varphi)$$

$$= r\sin\varphi\,(r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$$

$$= r\sin\varphi\ \sigma(\varphi,\theta)$$

$$\Rightarrow \|N_\sigma(\varphi,\theta)\| = |r\sin\varphi|\ \|\sigma(\varphi,\theta)\|$$

$$= r\sin\varphi\ r = r^2\sin\varphi$$

$\sin\varphi = 0 \iff \varphi = 0,\ \varphi = \pi$

$\Rightarrow \forall (\varphi,\theta) \in \text{int}(K)$: $\|N_\sigma(\varphi,\theta)\| \neq 0$

Quindi: la sup. è regolare

Per ogni $(\varphi,\theta) \in \text{int}(K)$:

$$n_\sigma(\varphi,\theta) = \frac{N_\sigma(\varphi,\theta)}{\|N_\sigma(\varphi,\theta)\|} = \frac{\cancel{r\sin\varphi}\;\sigma(\varphi,\theta)}{r^{\cancel{2}}\cancel{\sin\varphi}}$$

$$= \frac{\sigma(\varphi,\theta)}{r} = \underbrace{(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)}$$

è definito e continuo in tutto $\mathbb{R}^2$, quindi in tutto K (e non solo in $\mathrm{int}(K)$).