30 aprile 2024

Esempi (sullo jacobiano di funzioni composte)

$f(x,y) = (x+y,\ xy,\ xy^2)$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}^3$

$g(u,v,w) = uv^2w$ $\qquad g: \mathbb{R}^3 \to \mathbb{R}$

$h := g \circ f : \mathbb{R}^2 \to \mathbb{R}$

$$\forall (x,y) \in \mathbb{R}^2 : \quad h(x,y) = g(f(x,y)) = g(x+y,\ xy,\ xy^2)$$
$$= (x+y)(xy)^2(xy^2)$$
$$= (x+y)\, x^3y^4 = x^4y^4 + x^3y^5$$

$$J_h(x,y) = \begin{pmatrix} 4x^3y^4 + 3x^2y^5 & 4x^4y^3 + 5x^3y^4 \end{pmatrix}$$

$$J_g(u,v,w) = \begin{pmatrix} v^2w & 2uvw & uv^2 \end{pmatrix} \Rightarrow$$

$$J_g(f(x,y)) = J_g(x+y,\ xy,\ xy^2)$$
$$= \begin{pmatrix} (xy)^2(xy^2) & 2(x+y)(xy)(xy^2) & (x+y)(xy)^2 \end{pmatrix}$$
$$= \begin{pmatrix} x^3y^4 & 2(x+y)x^2y^3 & (x+y)x^2y^2 \end{pmatrix}$$

$f(x,y) = (x+y,\ xy,\ xy^2)$

$$J_g(f(x,y))\, J_f(x,y) =$$

$$\begin{pmatrix} x^3y^4 & 2x^3y^3 + 2x^2y^4 & x^3y^2 + x^2y^3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ y & x \\ y^2 & 2xy \end{pmatrix} =$$

$$\left(\underline{x^3y^4} + \underline{2x^3y^4} + 2x^2y^5 + \underline{x^3y^4} + x^2y^5 \quad \underline{x^3y^4} + 2x^4y^3 + \underline{2x^3y^4} + 2x^4y^3 + \underline{2x^3y^4} \right)$$

$$= \left(4x^3y^4 + 3x^2y^5 \quad 5x^3y^4 + 4x^4y^3 \right) \quad \checkmark$$

- $f(x) = (\underbrace{x+2}_{f_1(x)}, \underbrace{x^2+x}_{f_2(x)})$ $\qquad f: \mathbb{R} \to \mathbb{R}^2$

$g(u,v) = u^2 - 2v^3 \qquad g: \mathbb{R}^2 \to \mathbb{R}$

$h := g \circ f : \mathbb{R} \to \mathbb{R}$ t.c. $\forall x \in \mathbb{R}$

$$h(x) = g(f(x)) = g(x+2, x^2+x)$$
$$= (x+2)^2 - 2(x^2+x)^3$$

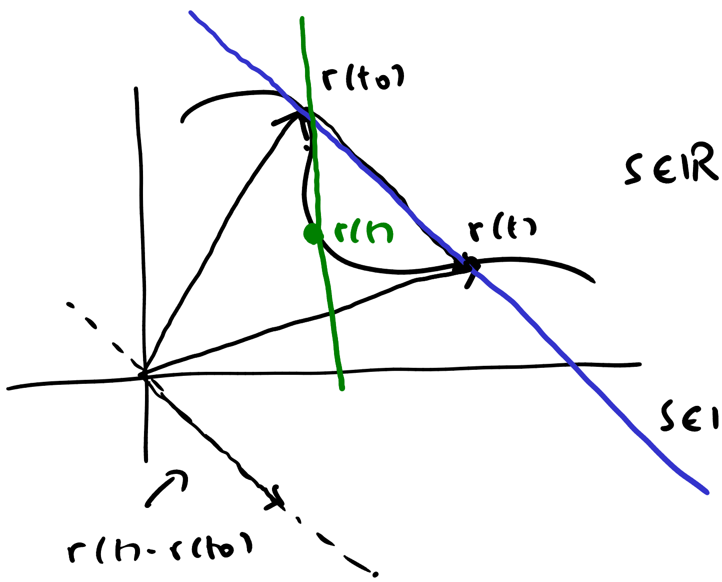
$$\Rightarrow h'(x) = 2(x+2) - 6(x^2+x)^2(2x+1)$$

Altro modo: $\quad g(u,v) = u^2 - 2v^3 \quad f(x) = (x+2, x^2+x)$

$$h'(x) = \frac{\partial g}{\partial u}(f(x))\, f_1'(x) + \frac{\partial g}{\partial v}(f(x))\, f_2'(x)$$

$$= 2 f_1(x) \cdot f_1'(x) - 6 (f_2(x))^2 f_2'(x)$$

$$= 2(x+2) \cdot 1 - 6(x^2+x)^2 (2x+1) \quad \checkmark$$



$$s \in \mathbb{R} \mapsto r(t_0) + s\,\frac{r(t) - r(t_0)}{t - t_0}$$

$$\Big\downarrow \; t \to t_0$$

$$s \in \mathbb{R} \mapsto r(t_0) + s\, r'(t_0)$$

Esempi (su curve regolari)

- $t \in [0,1] \mapsto \underbrace{x + t(y-x)}_{=: r(t)}$ con $x, y \in \mathbb{R}^n$, $x \neq y$

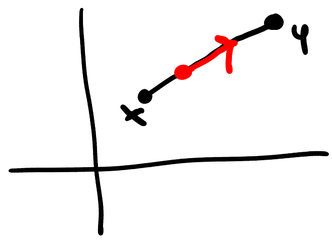
- $r \in C^1$? sì! $(r_i(t) = x_i + t(y_i - x_i))$

- non chiusa ✓

- $\forall t \in [0,1]$: $r'(t) = y - x \underset{\uparrow\, y \neq x}{\neq} 0$ ✓

Quindi: la curva è regolare e $\forall t \in [0,1]$:

$$T(t) := \frac{r'(t)}{\|r'(t)\|} = \frac{y-x}{\|y-x\|}$$



- $r(t) = (\underbrace{\cos t}_{\in C^1}, \underbrace{\sin t}_{\in C^1})$ $t \in [0, 2\pi]$ ✓

$r(0) = r(2\pi)$ curva chiusa

$\cos, \sin$ 2π-periodiche $\Rightarrow$ $r'(0) = r'(2\pi)$ ✓

$\forall t \in [0, 2\pi]$: $r'(t) = (-\sin t, \cos t)$

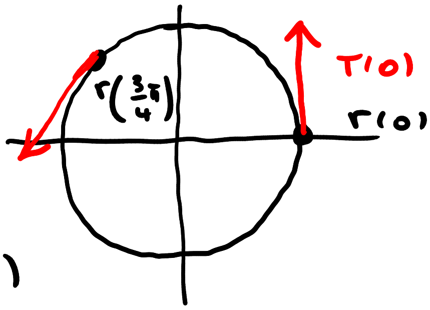
$r'(t) \neq (0,0)$ perché (come noto) seno e coseno non si annullano contemporaneamente.

In alternativa:

$$\|r'(t)\|^2 = \sin^2 t + \cos^2 t = 1 \neq 0$$

$$\Rightarrow \|r'(t)\| \neq 0 \quad \Rightarrow \quad r'(t) \neq (0,0)$$

Quindi: la curva è regolare.



Per ogni: $t \in [0, 2\pi]$:

$$T(t) = \frac{r'(t)}{\underbrace{\|r'(t)\|}_{=1}} = (-\sin t, \cos t)$$

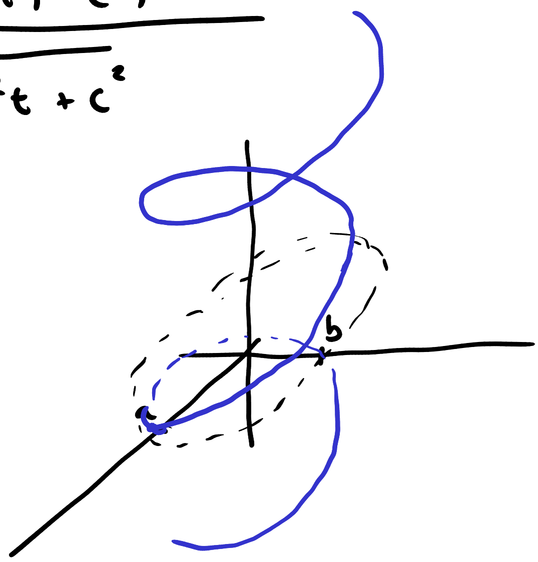
$$T\left(\frac{3}{4}\pi\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

- $r(t) = (\underbrace{a \cos t}_{C^1}, \underbrace{b \sin t}_{C^1}, \underbrace{ct}_{C^1})$ ✓ $a, b > 0$
  $t \in \mathbb{R}$ $c \neq 0$

  - non chiusa ✓
  - $\forall t \in \mathbb{R}$: $r'(t) = (-a \sin t, b \cos t, \underbrace{c}_{\neq 0}) \neq (0,0,0)$

  Quindi: curva regolare

  $$\forall t \in \mathbb{R}: \quad T(t) = \frac{(-a \sin t, b \cos t, c)}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t + c^2}}$$



- $r(t) = (\underbrace{t^3}_{C^1}, \underbrace{t^2}_{C^1})$ ✓, $t \in [-1, 1]$

  $r(-1) = (-1, 1)$, $r(1) = (1, 1)$ $\Rightarrow$ non chiusa ✓

$r'(t) = (3t^2, 2t) = (0,0) \iff t = 0$

$\Rightarrow$ curva <u>non</u> regolare

Oss: $\forall t \in [-1,1] \setminus \{0\}$:

$$T(t) = \frac{(3t^2, 2t)}{\sqrt{9t^4 + 4t^2}}$$

$$t \to 0: \quad T(t) \sim \left(\frac{3t^2}{2|t|}, \frac{\cancel{2}t}{\cancel{2}|t|} \right)$$

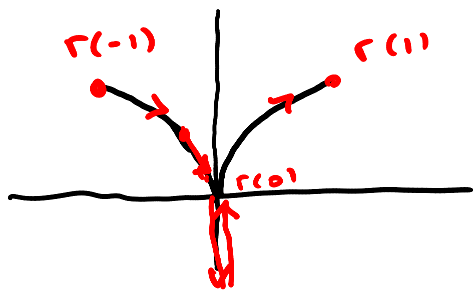
$\frac{3t^2}{2|t|} \to 0$ per $t \to 0^+$, $t \to 0^-$

$\frac{t}{|t|} \to 1$ per $t \to 0^+$; $\to -1$ per $t \to 0^-$

Scrivo le eq. parametriche:

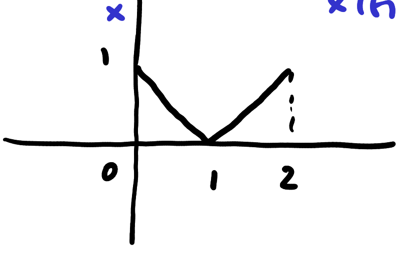
$$\begin{cases} x = t^3 \\ y = t^2 \end{cases} \quad t \in [-1,1]$$

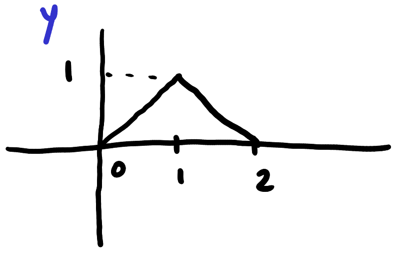
Da $x = t^3$ ricavo $t = x^{1/3}$; sostituisco in $y = t^2$ e ho $y = x^{2/3}$



Esempi (su curve regolari a tratti)

- $r(t) = (\underbrace{|t-1|}_{x(t)}, \underbrace{1 - |t-1|}_{y(t)})$ $t \in [0,2]$





$$r(t) = \begin{cases} (1-t,\ 1-1+t) & t \in [0,1] \\ (t-1,\ 2-t) & t \in (1,2] \end{cases}$$

$$[0,2] = [0,1] \cup [1,2]$$

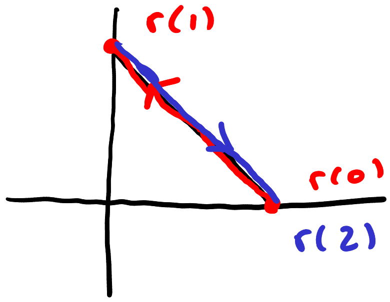
C^1 ✓

$\forall t \in [0,1)$: $r'(t) = (-1,1) \neq (0,0)$ ✓

$\forall t \in (1,2]$: $r'(t) = (1,-1) \neq (0,0)$

$\Rightarrow$ curva regolare a tratti.

Oss : $\left.\begin{array}{l} x = |t-1| \\ y = 1-|t-1| \end{array}\right| \Rightarrow y = 1-x$



- $r(t) = (\underbrace{t(t-1)}_{x(t)},\ \underbrace{t(t-1)(2t-1)}_{y(t)})\ ,\quad t \in \mathbb{R}$

x, y di classe C^1 ✓

$x(t) = t^2 - t \quad \Rightarrow \quad x'(t) = 2t-1$

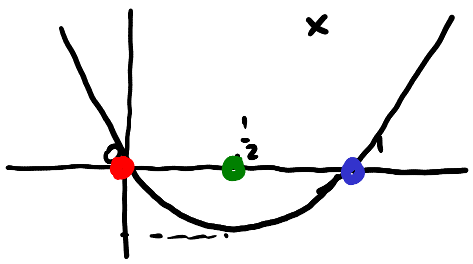
$y(t) = (t^2-t)(2t-1) = 2t^3 - 3t^2 + t \quad \Rightarrow \quad y'(t) = 6t^2 - 6t + 1$

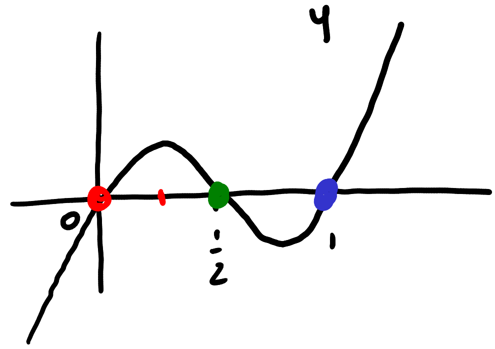
$$r'(t) = (0,0) \iff \begin{cases} 2t-1 = 0 & t = \frac{1}{2} \\ 6t^2 - 6t + 1 = 0 & 6 \cdot \frac{1}{4} - 6 \cdot \frac{1}{2} + 1 \neq 0 \end{cases}$$

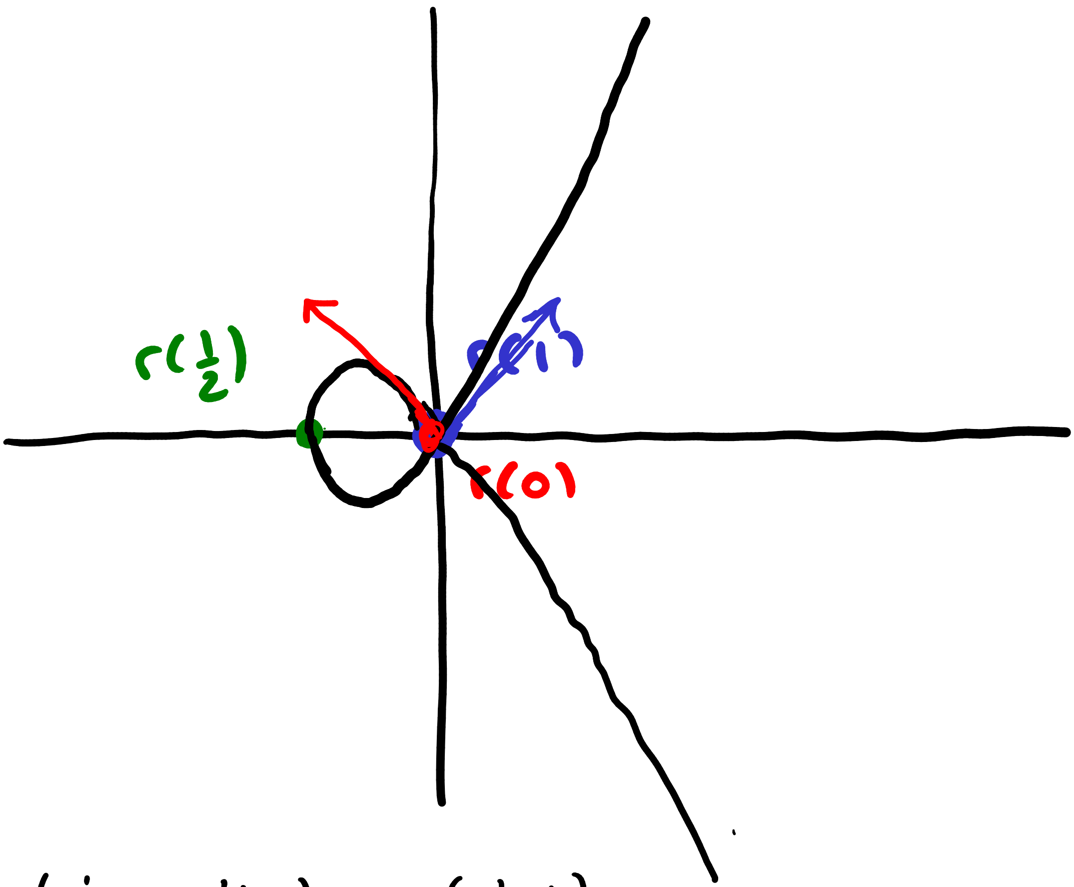
$\Rightarrow \quad r'(t) \neq (0,0) \quad \forall t \in \mathbb{R}$

$\Rightarrow$ curva regolare

Disegno il sostegno :



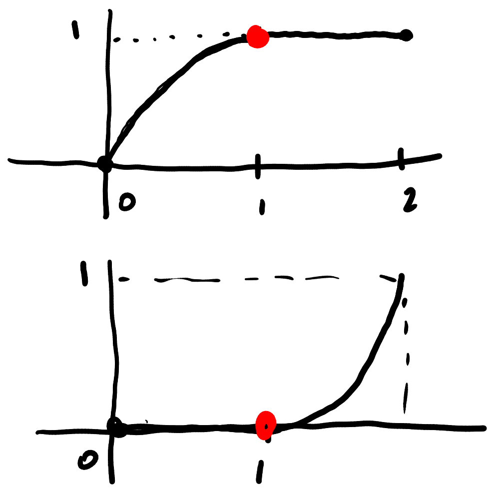




$$T(0) = \frac{(x'(0), y'(0))}{\sqrt{x'(0)^2 + y'(0)^2}} = \frac{(-1,1)}{\sqrt{2}}$$

$$T(1) = \frac{(x'(1), y'(1))}{\sqrt{x'(1)^2 + y'(1)^2}} = \frac{(1,1)}{\sqrt{2}}$$

- $r(t) = \begin{cases} (2t - t^2,\ 0) & t \in [0,1) \\ (1,\ (t-1)^2) & t \in [1,2] \end{cases}$



$x(t) = \begin{cases} 2t - t^2 & t \in [0,1) \\ 1 & t \in [1,2] \end{cases}$

$x \in C^1$ ✓

$y(t) = \begin{cases} 0 & t \in [0,1) \\ (t-1)^2 & t \in [1,2] \end{cases}$

$y \in C^1$ ✓

$r'(t) = (0,0) \iff t = 1$

Curva regolare a tratti.