18 aprile 2024

Chiarimento (richiesto dal pubblico)

$r: I \to \mathbb{R}^n$, $r(I) = \gamma$

(γ, r) semplice $\overset{def}{\iff}$ $\forall\, t_1, t_2 \in I$, con $t_1 \neq t_2$ e almeno uno interno a I:

$$r(t_1) \neq r(t_2)$$

Per stabilire se la curva è semplice:
prendo $t_1, t_2 \in I$ e studio

$$r(t_1) = r(t_2) \qquad \odot$$

Se dimostro che $\odot$ è soddisfatta solo se $t_1 = t_2$ oppure $t_1 \neq t_2$ ma t_1 e t_2 sono entrambi estremi di I

allora posso dire che la curva è semplice.

Es: $r(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

Prendo $t_1, t_2 \in [0, 2\pi]$ e suppongo

$$r(t_1) = r(t_2) \iff$$

$$\begin{cases} \cos t_1 = \cos t_2 \implies t_1 = t_2 \text{ opp. } t_1 = 2\pi - t_2 \\ \sin t_1 = \sin t_2 \implies t_1 = t_2 \text{ opp. } t_1 = \pi - t_2 \end{cases}$$

$\implies 2\pi - t_2 = \pi - t_2$ assurdo! se $t_1, t_2 \notin \{0, 2\pi\}$

Es: $r(t) = \left((t+1)^2,\ t^2(t+2)\right) \quad t\in[-2,1]$

$r(t_1) = r(t_2)$

$$\begin{cases} (t_1+1)^2 = (t_2+1)^2 \\ t_1^2(t_1+2) = t_2^2(t_2+2) \end{cases} \longrightarrow \begin{array}{l} \Rightarrow t_1 = t_2 \\ t_1+1 = t_2+1 \quad \text{opp.} \\ t_1+1 = -(t_2+1) \\ \boxed{t_1+t_2 = -2} \end{array}$$

....

Verifico che le funzioni costanti sono differenziabili:

$c\in\mathbb{R} \quad f:\mathbb{R}^n\to\mathbb{R}$ t.c. $f(x) = c \quad \forall x\in\mathbb{R}^n$

Fisso $\bar{x}\in\mathbb{R}^n$; per $h\in\mathbb{R}^n\setminus\{0\}$ valuto

$$R(h) := \frac{f(\bar{x}+h) - f(\bar{x}) - \underbrace{0}_{=L_{\bar{x}}(h)}}{\|h\|} = \frac{c-c}{\|h\|} = 0$$

$\Rightarrow \exists \lim_{h\to 0} R(h) = 0$ $\square$

Verifico che le funzioni lineari sono differenziabili

$f:\mathbb{R}^n\to\mathbb{R}$ lineare

$\bar{x}\in\mathbb{R}^n$

$\forall h\in\mathbb{R}^n\setminus\{0\}$ valuto

$$R(h) := \frac{f(\bar{x}+h) - f(\bar{x}) - \overbrace{f(h)}^{=L_{\bar{x}}(h)}}{\|h\|}$$

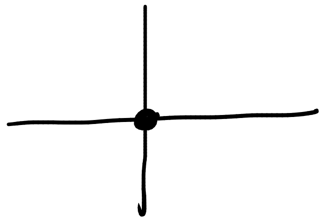
f lineare

$$\overset{f\ \text{lineare}}{=} \frac{f(\bar{x}) + f(h) - f(\bar{x}) - f(h)}{\|h\|} = 0$$

$$\Rightarrow \exists \lim_{h \to 0} R(h) = 0 \qquad \square$$

Esempi sulla differenziabilità

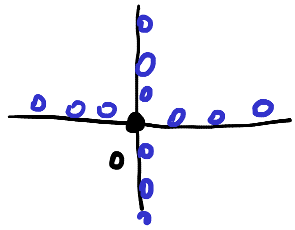
- $f(x,y) = \sqrt{x^2+y^2}$ $\quad (0,0)$



Già studiata:
non è derivabile parzialmente
né rispetto a x, né rispetto a y, in $(0,0)$

$\Rightarrow$ non è differenziabile. $\square$

- $f(x,y) = |xy|$ $\quad (0,0)$



$\mathrm{dom}(f) = \mathbb{R}^2$

Verifico se è deriv. parz. rispetto a x:

$$\lim_{t \to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0-0}{t} = 0 =: \frac{\partial f}{\partial x}(0,0)$$

Analogamente: $\frac{\partial f}{\partial y}(0,0) = 0$

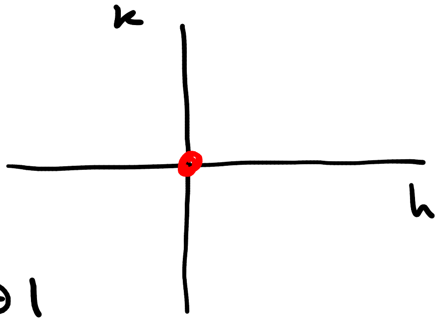
Considero il rapporto incrementale.
Per ogni: $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$R(h,k) := \frac{f((0,0)+(h,k)) - f(0,0) - \nabla f(0,0) \cdot (h,k)}{\|(h,k)\|}$$

$$= \frac{f(h,k) - \overbrace{f(0,0)}^{=0} - \left(\overbrace{\frac{\partial f}{\partial x}(0,0)}^{=0} h + \overbrace{\frac{\partial f}{\partial y}(0,0)}^{=0} k \right)}{\sqrt{h^2+k^2}}$$

$$= \frac{|hk|}{\sqrt{h^2+k^2}}$$

Domanda: $\quad \exists \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} = 0 \quad ??$



Suggerimento dal pubblico:

$h = \rho\cos\theta, \quad k = \rho\sin\theta$

$$R(\rho\cos\theta, \rho\sin\theta) = \frac{|\rho^2\cos\theta\sin\theta|}{\sqrt{\rho^2}}$$

$$= \rho \underbrace{|\cos\theta\sin\theta|}_{\leq 1}$$

$$\Rightarrow \lim_{\rho\to 0^+} \sup_{\theta\in[0,2\pi)} R(\rho\cos\theta, \rho\sin\theta) = 0$$

$$\Rightarrow \exists \lim_{(h,k)\to(0,0)} R(h,k) = 0$$

$\Rightarrow f$ è differenziabile in $(0,0)$

con $df_{(0,0)} \equiv 0$

$$\uparrow$$

$$df_{(0,0)}(h,k) = \nabla f(0,0)\cdot(h,k) = 0$$

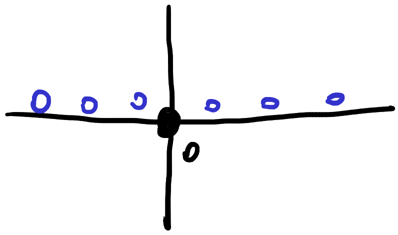
In alternativa alle coordinate polari:

$$0 \leq |R(h,k)| = \frac{|hk|}{\sqrt{h^2+k^2}} = \underbrace{\frac{|h|}{\sqrt{h^2+k^2}}}_{\leq 1} \underbrace{|k|}_{\to 0} \quad (h,k)\to(0,0)$$

$$\overset{TCo}{\Rightarrow} R(h,k) \to 0 \quad \text{per} \quad (h,k)\to(0,0) \qquad \square$$

$$\bullet \quad f(x,y) = \begin{cases} \dfrac{x^2 y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \qquad \underline{(0,0)}$$

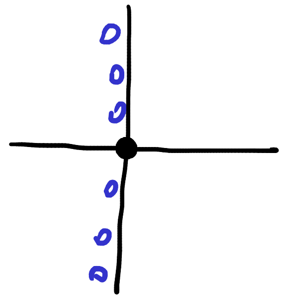
$\mathrm{dom}(f) = \mathbb{R}^2$



Oss: $\quad f(x,0) = 0 \quad \forall x \quad \Rightarrow \quad \exists \dfrac{\partial f}{\partial x}(x,0) = 0$

$$\Rightarrow \quad \exists \frac{\partial f}{\partial x}(0,0) = 0$$

$f(0,y) = 0 \quad \forall y \quad \Rightarrow \quad \exists \dfrac{\partial f}{\partial y}(0,y) = 0$



$$\Rightarrow \quad \exists \frac{\partial f}{\partial y}(0,0) = 0$$

Per ogni: $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$R(h,k) = \frac{f(h,k) - \overset{=0}{f(0,0)} - \overset{=(0,0)}{\nabla f(0,0)} \cdot (h,k)}{\sqrt{h^2+k^2}}$$

$$= \frac{\dfrac{h^2k^3}{h^2+k^2}}{\sqrt{h^2+k^2}} = \frac{h^2k^3}{h^2+k^2} \cdot \frac{1}{\sqrt{h^2+k^2}}$$

Oss:

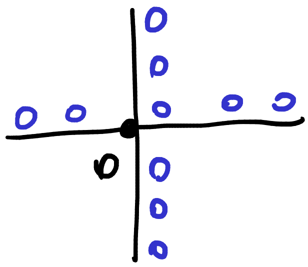
$$0 \leq |R(h,k)| = \underbrace{\frac{h^2}{h^2+k^2}}_{\leq 1} \cdot \underbrace{\frac{|k|}{\sqrt{h^2+k^2}}}_{\leq 1} \cdot k^2 \to 0 \quad (h,k) \to (0,0)$$

$$\overset{TCO}{\Rightarrow} \quad \exists \lim_{(h,k)\to(0,0)} R(h,k) = 0$$

$\Rightarrow f$ è differenziabile in $(0,0)$ con $df_{(0,0)} \equiv 0$. $\square$

$$\bullet \quad f(x,y) = \begin{cases} \dfrac{x^4 y^2}{(x^4+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$(0,0)$

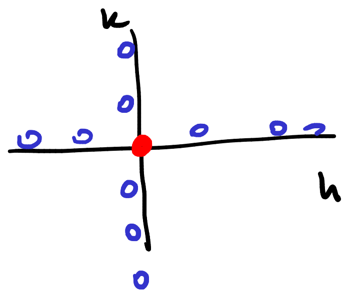


$$\Rightarrow \quad \exists \ \frac{\partial f}{\partial x}(0,0) = 0, \qquad \exists \ \frac{\partial f}{\partial y}(0,0) = 0$$

Per ogni $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$R(h,k) = \frac{f(h,k) - \underbrace{f(0,0)}_{=0} - \nabla f(\overset{(0,0)}{0,0}) \cdot (h,k)}{\sqrt{h^2+k^2}}$$

$$= \frac{h^4 k^2}{(h^4+k^2)^2} \cdot \frac{1}{\sqrt{h^2+k^2}}$$



$\|$

$$0 \leq \underbrace{\left(\frac{h^4}{h^4+k^2}\right)}_{\leq 1} \cdot \underbrace{\left(\frac{k^2}{h^4+k^2}\right)}_{\leq 1} \cdot \underbrace{\left(\frac{1}{\sqrt{h^2+k^2}}\right)}_{\substack{\to +\infty \\ (h,k)\to(0,0)}}$$

??!

$$R(h,h) = \frac{h^6}{(h^4+h^2)^2} \cdot \frac{1}{\sqrt{2h^2}} \sim \frac{h^6}{h^4} \cdot \frac{1}{\sqrt{2h^2}} = \frac{h^2}{\sqrt{2h^2}} \xrightarrow[h\to 0]{} 0 \quad ??$$

$h^4 = o(h^2)$

$$R(h,h^2) = \frac{h^4 h^4}{(h^4+h^4)^2} \cdot \frac{1}{\sqrt{h^2+h^4}} = \frac{\cancel{h^8}}{4\cancel{h^8}} \cdot \underbrace{\left(\frac{1}{\sqrt{h^2+h^4}}\right)}_{\to +\infty} \xrightarrow[h\to 0]{} +\infty$$

Quindi: $R(h,k)$ non tende a 0

per $(h,k) \to (0,0)$

$\Rightarrow$ f non è diff. in $(0,0)$. □