16 aprile 2024

Esempi di derivate parziali

- $f(x,y) = 8xy + 5x^4y^2 - y^3$

funzione polinomiale ; $\mathrm{dom}(f) = \mathbb{R}^2$

Per ogni $(x,y) \in \mathbb{R}^2$:

$$\frac{\partial f}{\partial x}(x,y) = 8y + 20x^3y^2 + 0 = 8y + 20x^3y^2$$

$$8x\underbrace{y}_{\text{cost.}} + 5x^4\underbrace{y^2}_{\text{cost.}} - \underbrace{y^3}_{\text{cost.}}$$

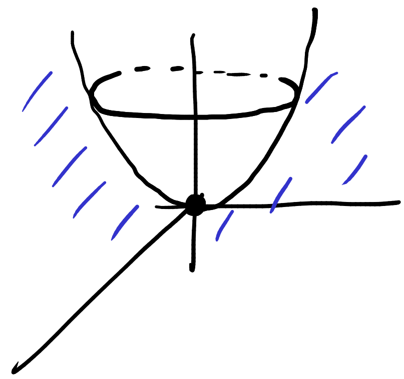
$$\frac{\partial f}{\partial y}(x,y) = 8x + 10x^4y - 3y^2$$

$$8\underbrace{x}_{\text{cost}}y + 5\underbrace{x^4}_{\text{cost}}y^2 - y^3$$

- $f(x,y,z) = \ln(x^2+y^2-z)$

$\underline{\mathrm{dom}(f)} = \{(x,y,z) \mid x^2+y^2-z>0\}$
$= \{(x,y,z) \mid z < x^2+y^2\}$

aperto



f composta di:

$(x,y,z) \mapsto x^2+y^2-z$ polinomiale

$\Rightarrow$ derivabile rispetto a ciascuna variabile

$t \mapsto \ln(t)$ derivabile

Quindi: f derivabile rispetto a ciascuna variabile

$\forall (x,y,z) \in \mathrm{dom}(f)$:

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{1}{x^2+y^2-z} \cdot 2x$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{1}{x^2+y^2-z} \cdot 2y$$

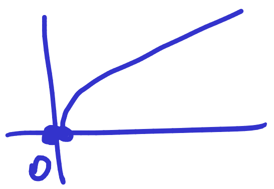
$$\frac{\partial f}{\partial z}(x,y,z) = \frac{1}{x^2+y^2-z} (-1)$$

- $f(x,y) = \sqrt{x^2+y^2}$ $\quad (= \|(x,y)\|)$

$\mathrm{dom}(f) = \mathbb{R}^2$

f è composta di:

$(x,y) \mapsto x^2+y^2$ $\quad$ polinomiale ✓



$t \mapsto \sqrt{t}$ $\quad$ derivabile per $t>0$

non deriv. in $t=0$

Quindi: $f|_{\mathbb{R}^2 \setminus \{(0,0)\}}$ è derivabile parzialmente rispetto a entrambe le variabili e per ogni $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

In $(0,0)$ non posso applicare la regola di derivazione della funzione composta (perché $0^2+0^2=0$ e $t \mapsto \sqrt{t}$ non è derivabile in $t=0$); uso la definizione:

$$\lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t} =$$

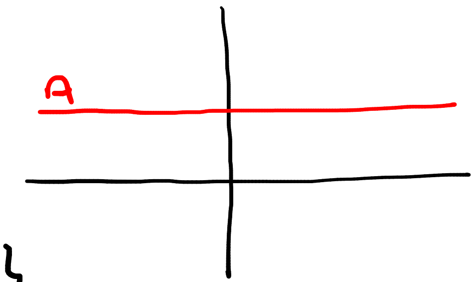
$$\lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{\sqrt{t^2+0^2}-\sqrt{0^2+0^2}}{t}$$

$$= \lim_{t\to 0} \frac{\sqrt{t^2}}{t} = \lim_{t\to 0} \frac{|t|}{t}$$ non esiste!

Quindi: in $(0,0)$ f <u>non</u> è derivabile parzialmente rispetto a x.

Analogamente per y.

- $f(x,y) = x^2(2+|y-1|)$



$\mathrm{dom}(f) = \mathbb{R}^2$

Definisco $A := \{(x,y) \mid y=1\}$

In $\mathbb{R}^2 \setminus A$: f è composta di funzioni derivabili, quindi è derivabile parzialmente rispetto a x e y e per ogni $(x,y) \in \mathbb{R}^2 \setminus A$:

$$\frac{\partial f}{\partial x}(x,y) = 2x(2+|y-1|)$$

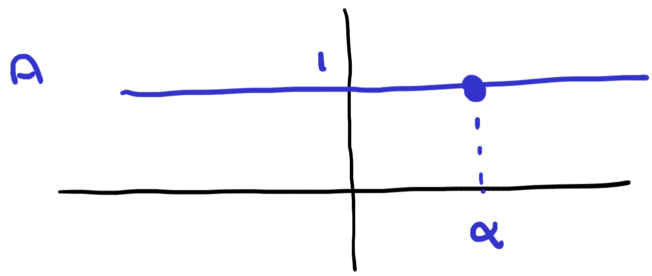
$g(t) = |t|$

$\forall t \neq 0: \quad g'(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} = \operatorname{sign}(t) = \frac{t}{|t|} = \frac{|t|}{t}$

$$\frac{\partial f}{\partial y}(x,y) = x^2 \left(0 + \operatorname{sign}(y-1) \cdot 1 \right)$$

$$= x^2 \operatorname{sign}(y-1)$$

Studio la derivabilità nei punti di A.



Fisso $\alpha \in \mathbb{R}$ e considero $(\alpha, 1)$

Osservo che

$f|_A (x,y) = 2x^2$ derivabile rispetto a x, per ogni x

$\left(\Rightarrow \exists \frac{\partial f}{\partial x}(x, \underset{=1}{y}) = 4x \right.$

$\left. \Rightarrow \exists \frac{\partial f}{\partial x}(\alpha,1) = 4\alpha \right)$

Con la definizione:

$$\lim_{t \to 0} \frac{f(\alpha + t, 1) - f(\alpha, 1)}{t}$$

$$= \lim_{t \to 0} \frac{(\alpha+t)^2 (2 + |1-1|) - \alpha^2 (2 + |1-1|)}{t}$$

$$= \lim_{t \to 0} \frac{2(\alpha+t)^2 - 2\alpha^2}{t} = \lim_{t \to 0} 2\, \frac{\cancel{\alpha^2} + 2\alpha t + t^2 - \cancel{\alpha^2}}{t}$$

$$= 4\alpha =: \frac{\partial f}{\partial x}(\alpha, 1)$$

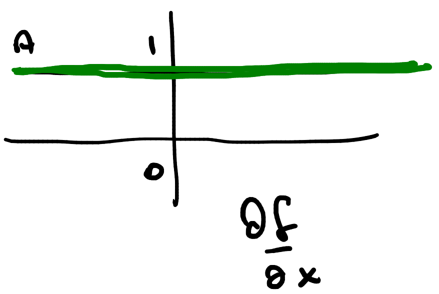
Con la definizione:

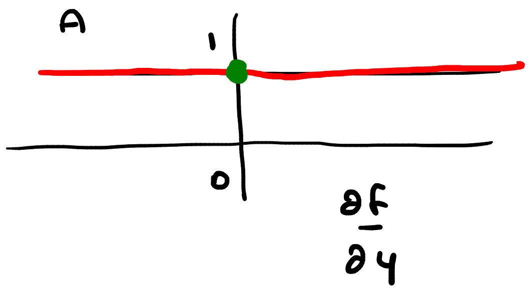
$$\lim_{t\to 0} \frac{f(\alpha, 1+t) - f(\alpha,1)}{t} =$$

$$\lim_{t\to 0} \frac{\alpha^2(2+|1+t-1|) - \alpha^2(2+|1-1|)}{t} =$$

$$\lim_{t\to 0} \frac{\alpha^2(2+|t|) - 2\alpha^2}{t} =$$

$$\lim_{t\to 0} \frac{\alpha^2 |t|}{t} = \begin{cases} \alpha \neq 0 & \text{NON ESISTE!} \\ \alpha = 0 & 0 =: \dfrac{\partial f}{\partial y}(0,1) \end{cases}$$

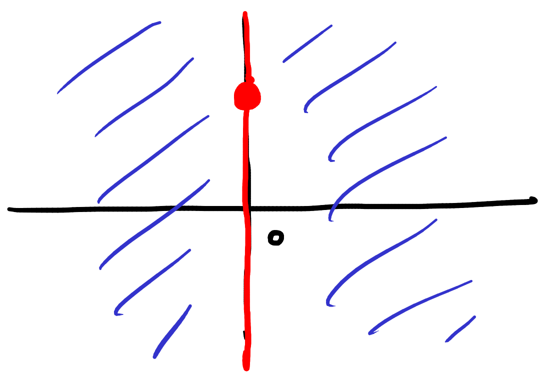




• $f(x,y) = \begin{cases} \boxed{x(y+3)} & x \neq 0 \\ \boxed{y^2} & x = 0 \end{cases}$ $\quad \mathrm{dom}(f) = \mathbb{R}^2$

$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,y) \mid y \in \mathbb{R}\}$:

f è deriv. parzialmente rispetto a x e a y



con

$$\frac{\partial f}{\partial x}(x,y) = y+3, \quad \frac{\partial f}{\partial y}(x,y) = x$$

Nei punti $(0,\beta)$, con $\beta\in\mathbb{R}$, uso la definizione.

$$\forall t\neq 0: \quad \frac{f(0+t,\beta)-f(0,\beta)}{t} = \frac{f(t,\beta)-f(0,\beta)}{t}$$

$$= \frac{t(\beta+3)-\beta^2}{t} = \beta+3-\frac{\beta^2}{t} \quad (*)$$

Oss: $\beta\neq 0$: $\quad \frac{\beta^2}{t} \to \pm\infty$ se $t\to 0^{\pm}$

$\Rightarrow$ Per $\beta\neq 0$: f non è derivabile parzial. rispetto a x in $(0,\beta)$

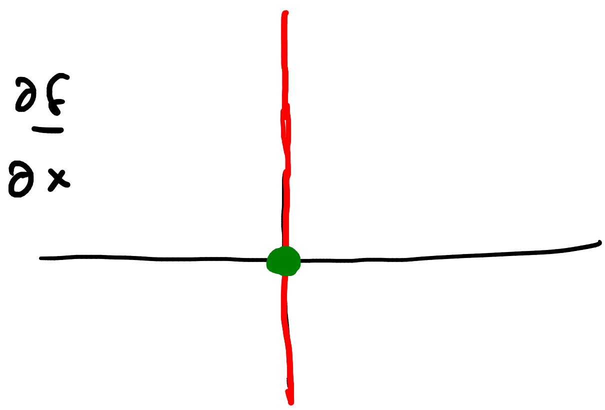
Se $\beta = 0$: $\quad (*) \equiv 3 \quad \Rightarrow \quad \exists\ \frac{\partial f}{\partial x}(0,0) = 3$

Per la derivabilità rispetto a y:

$$\forall t\neq 0: \quad \frac{f(0,\beta+t)-f(0,\beta)}{t} = \frac{(\beta+t)^2-\beta^2}{t}$$

$$= \frac{\beta^2+2\beta t+t^2-\beta^2}{t} = \frac{2\beta t+t^2}{t} \xrightarrow[t\to 0]{} 2\beta$$

$$\Rightarrow \quad \exists\ \frac{\partial f}{\partial y}(0,\beta) = 2\beta \quad \forall \beta\in\mathbb{R}.$$



Esempio sul gradiente

$f(x,y,z) = \ln(x^2+y^2-z)$ $\qquad$ $(\bar{x},\bar{y},\bar{z}) = (1,1,-1)$

$\in \mathrm{dom}(f)$
$\uparrow$
aperto

Già visto:

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{2x}{x^2+y^2-z}$$

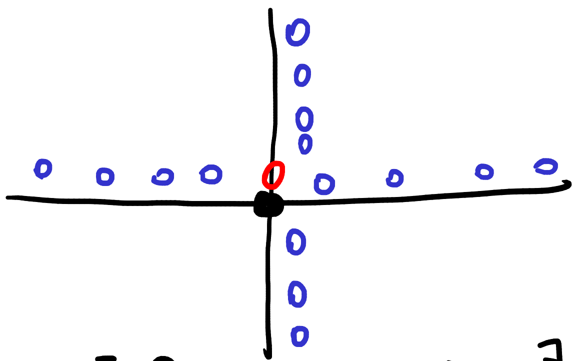
$$\frac{\partial f}{\partial y}(x,y,z) = \frac{2y}{x^2+y^2-z}$$

$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{1}{x^2+y^2-z}$$

$$\nabla f(1,1,-1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

Esempio di funzione non continua derivabile in qualsiasi direzione.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



$f|_{\{y=0\}} \equiv 0 \implies \exists \dfrac{\partial f}{\partial x}(0,0) = 0$

$f|_{\{x=0\}} \equiv 0 \implies \exists \dfrac{\partial f}{\partial y}(0,0) = 0$

Fisso $v \in \mathbb{R}^2$ con $v_1^2 + v_2^2 = 1$ e $v_1, v_2 \neq 0$
(già considerati)

$\forall t \neq 0$:

$$\frac{f((0,0)+t(v_1,v_2)) - f(0,0)}{t} = \frac{f(tv_1,tv_2) - f(0,0)}{t}$$

$$= \frac{\frac{(tv_1)^2(tv_2)}{(tv_1)^4+(tv_2)^2} - 0}{t} = \frac{t^3v_1^2v_2}{t^4v_1^4+t^2v_2^2}\cdot\frac{1}{t}$$

$$= \frac{t^3v_1^2v_2}{t^3(t^2v_1^4+v_2^2)} = \frac{v_1^2v_2}{t^2v_1^4+v_2^2} \xrightarrow[t\to 0]{} \frac{v_1^2v_2}{v_2^2} = \frac{v_1^2}{v_2}$$

$$\Rightarrow \exists \frac{\partial f}{\partial v}(0,0) = \frac{v_1^2}{v_2} \qquad \square$$

Verifico che $\pi_1, \dots, \pi_n$ sono linearmente indipendenti in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$

Prendo $c_1, \dots, c_n \in \mathbb{R}$ e suppongo che $c_1\pi_1 + \dots + c_n\pi_n$ sia la funzione costante di valore 0

di $\mathbb{R}^n$ in $\mathbb{R}$

Fisso $j \in \{1,\dots,n\}$. Dato che $\sum_{i=1}^n c_i\pi_i$ è la funzione costante di valore 0, ho:

$$0 = \left(\sum_{i=1}^n c_i\pi_i\right)(e_j) \underset{\substack{\text{def. di}\\ \text{funz. comb. lin.}}}{=} \sum_{i=1}^n c_i \underbrace{\pi_i(e_j)}_{=\delta_{ij} = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}} = c_j$$

Dunque: $c_1 = \dots = c_n = 0 \Rightarrow$
$\pi_1, \dots, \pi_n$ lin. indip.

Prendo $x \in \mathbb{R}^n$: $x = \sum_{i=1}^{n} \underbrace{x_i}_{=\pi_i(x)} e_i$

$$L(x) = L\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i L(e_i)$$

L lineare

$$= \sum_{i=1}^{n} L(e_i)\, x_i = \sum_{i=1}^{n} L(e_i)\, \pi_i(x)$$

$$= \left(\sum_{i=1}^{n} L(e_i)\pi_i\right)(x)$$

$$\Rightarrow L = \sum_{i=1}^{n} L(e_i)\, \pi_i \qquad \square$$