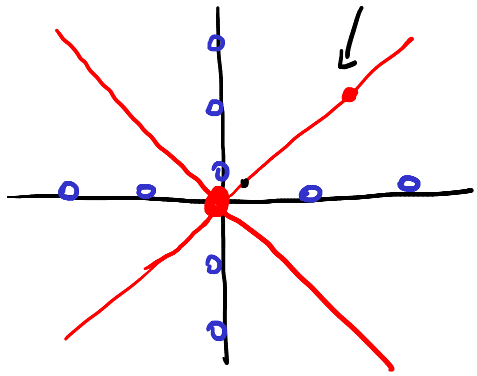
12 aprile 2024

$\nexists$ lim



Riprendo $f(x,y) = \dfrac{x^2 y}{x^2 - y^2}$

In sospeso: limite per $(x,y) \to (0,0)$

Oss: se esiste, è uguale a 0

Provo: $y = mx \qquad m \neq \pm 1$

$$f(x, mx) = \frac{x^2 \, mx}{x^2 - m^2 x^2} = \frac{mx}{1-m^2} \xrightarrow[x \to 0]{} 0 \qquad ??$$

Provo con coord. polari: $x = \rho \cos\theta, \quad y = \rho \sin\theta$

$$g(\rho, \theta) := f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^2 \cos^2\theta \, \rho \sin\theta}{\rho^2\cos^2\theta - \rho^2 \sin^2\theta}$$

$$= \rho \frac{\cos^2\theta \sin\theta}{\cos^2\theta - \sin^2\theta}$$

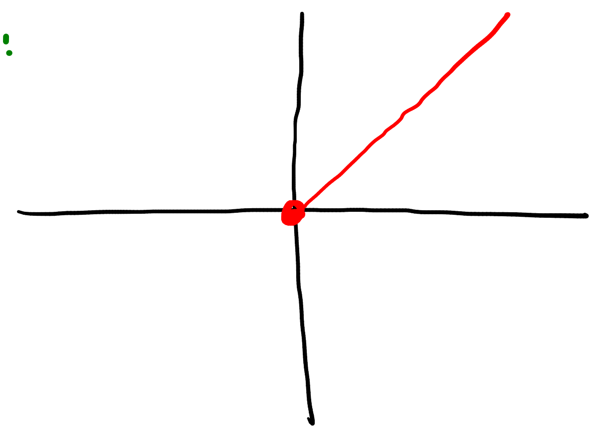
non è limitata!
(problema: $\theta \to \frac{\pi}{4}$
$\theta \to \frac{3}{4}\pi$)

Provo a costruire una successione (x_k, y_k) di elementi di dom(f) tc. $(x_k, y_k) \to (0,0)$ e $f(x_k, y_k) \not\to 0$

Suggerimento dal pubblico:

$\forall k \geq 1$:

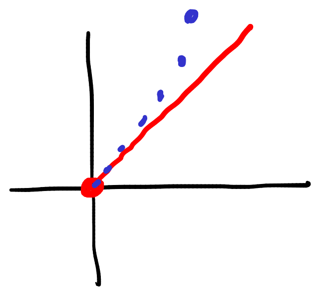
$x_k = \frac{1}{k}, \quad y_k = \frac{1}{k^2}$



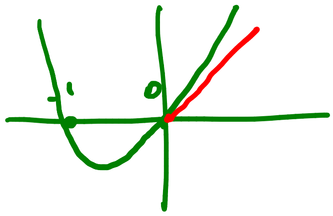
Valuto:

$$f(x_k, y_k) = \frac{\frac{1}{k^2} \cdot \frac{1}{k^2}}{\frac{1}{k^2} - \frac{1}{k^4}} = \frac{\overbrace{\frac{1}{k^2}}^{\to 0}}{\underbrace{1 - \frac{1}{k^2}}_{\to 1}} \longrightarrow 0 \qquad ???$$

Provo: $x_k = \frac{1}{k}$, $\quad y_k = \frac{1}{k} + \frac{1}{k^2}$



[$\varphi(x) = x + x^2$



]

Valuto:

$$f(x_k, y_k) = \frac{\frac{1}{k^2}\left(\frac{1}{k} + \frac{1}{k^2}\right)}{\frac{1}{k^2} - \left(\frac{1}{k} + \frac{1}{k^2}\right)^2} = \frac{\frac{1}{k^3}\left(1 + \frac{1}{k}\right)}{\cancel{\frac{1}{k^2}} - \cancel{\frac{1}{k^2}} - \frac{2}{k^3} - \frac{1}{k^4}}$$

$$= \frac{\cancel{\frac{1}{k^3}}\left(1 + \frac{1}{k}\right)}{\cancel{\frac{1}{k^3}}\left(-2 - \frac{1}{k}\right)} \xrightarrow[k \to +\infty]{} -\frac{1}{2} \neq 0$$

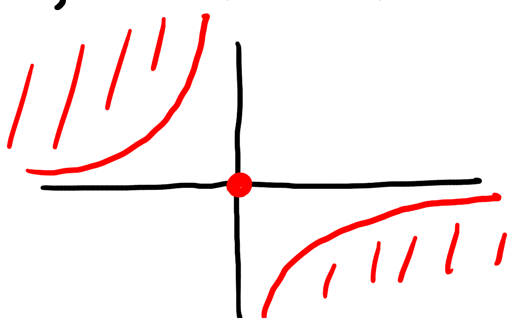
$$\Rightarrow \nexists \lim_{(x,y) \to (0,0)} f(x,y) \qquad \square$$

- $f(x,y) = \dfrac{\ln(1 + xy)}{x^2 + y^2}$ $\qquad \begin{cases} 1 + xy > 0 \\ x^2 + y^2 \neq 0 \end{cases}$

$$\operatorname{dom}(f) = \{(x,y) \mid 1 + xy > 0, \ x^2 + y^2 \neq 0\}$$

[$1 + xy = 0$

$xy = -1 \qquad y = -\frac{1}{x}$]



$$
\begin{cases}
xy > -1 \\
\text{se } x > 0: \quad y > -\frac{1}{x} \\
\text{se } x < 0: \quad y < -\frac{1}{x}
\end{cases}
$$

f è continua (composta di funz. continue)

$\Rightarrow$ limiti significativi:

$(x,y) \to (a, -\frac{1}{a}) \qquad a \neq 0$

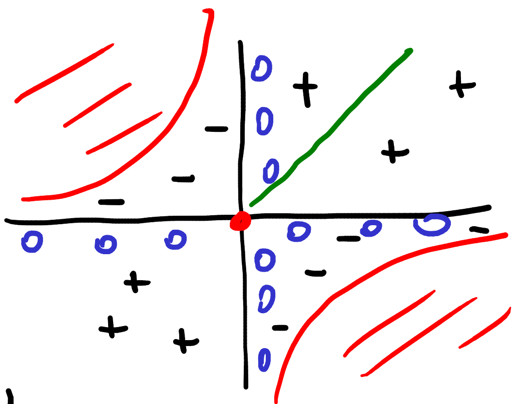
$(x,y) \to (0,0)$

$\|(x,y)\| \to +\infty$

Fisso $a \neq 0$ e considero $(a, -\frac{1}{a})$

$$\lim_{(x,y)\to(a,-\frac{1}{a})} f(x,y) = \lim_{(x,y)\to(a,-\frac{1}{a})} \frac{\overbrace{\ln(\underbrace{1+xy}_{\to 0})}^{\to -\infty}}{\underbrace{x^2+y^2}_{\to a^2+\frac{1}{a^2}>0}} = -\infty$$

Considero $(0,0)$:



se esiste, il limite per $(x,y)\to(0,0)$ è uguale a 0.

Oss: $\underline{\underline{B}} := \{(x,x) \mid x > 0\}$

$$\lim_{(x,y)\to(0,0)} f_{|_B}(x,y) = \lim_{x\to 0^+} f(x,x) = \lim_{x\to 0^+} \frac{\ln(1+x^2)}{2x^2}$$

$\ln(1+t) \underset{t\to 0}{\sim} t \;\to\; = \frac{1}{2} \neq 0$

$\Rightarrow$ il limite non esiste.

Considero ora $\|(x,y)\| \to +\infty$.

$f|_B \to 0$, il che conferma la congettura che $f \to 0$ per $\|(x,y)\| \to +\infty$

Verifico che, posto $C := \{(x,y) \mid xy > 0\}$, si ha:

$$\lim_{\|(x,y)\| \to +\infty} f|_C(x,y) = 0$$

Oss: $\forall (x,y) \in C$:

$$0 < xy = |xy| \leq \frac{x^2+y^2}{2} \Rightarrow$$

$$1+xy \leq 1+\frac{x^2+y^2}{2} \Rightarrow$$

ln crescente

$$0 < f|_C(x,y) = \frac{\ln(1+xy)}{x^2+y^2} \leq \underbrace{\frac{\ln\left(1+\frac{x^2+y^2}{2}\right)}{x^2+y^2}}_{\to 0}$$

$$f|_C(x,y) \xrightarrow{\|(x,y)\| \to +\infty} 0$$

$$\lim_{\|(x,y)\| \to +\infty} \frac{\ln\left(1+\frac{x^2+y^2}{2}\right)}{x^2+y^2} = \lim_{t \to +\infty} \frac{\ln\left(1+\frac{t}{2}\right)}{t} = 0$$

TCo

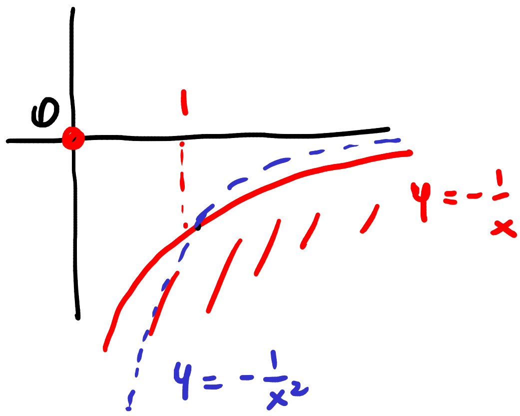
$$\Rightarrow \lim_{\|(x,y)\| \to +\infty} f|_C(x,y) = 0.$$

[Vincolo: $-1 < x_k y_k < 0$]

Suggerimento dal pubblico:

Provo:

$$x_k = k, \quad y_k = -\frac{1}{k^2}$$



$$x_k y_k = -\frac{1}{k} \quad \begin{array}{ll} <0 & \checkmark \\ >-1 & \forall k \geq 2 \end{array}$$

Valuto:

$$f(x_k, y_k) = \frac{\ln(1 + x_k y_k)}{x_k^2 + y_k^2} = \frac{\overbrace{\ln\left(1 - \frac{1}{k}\right)}^{\to 0}}{\underbrace{k^2 + \frac{1}{k^4}}_{\to +\infty}} \xrightarrow[k \to +\infty]{} 0 \quad ??$$

Ulteriore suggerimento dal pubblico

$$x_k = k, \qquad y_k = -\frac{1}{k} + \frac{1}{k^2} \in (-1, 0) \quad k \geq 2$$

Valuto:

$$f(x_k, y_k) = \frac{\ln\left(1 - 1 + \frac{1}{k}\right)}{k^2 + \left(-\frac{1}{k} + \frac{1}{k^2}\right)^2} = \frac{\overbrace{\ln\left(\overbrace{\tfrac{1}{k}}^{\to 0}\right)}^{\to -\infty}}{\underbrace{k^2 + \left(-\frac{1}{k} + \frac{1}{k^2}\right)^2}_{\to +\infty}}$$

$$\sim \frac{\ln\left(\frac{1}{k}\right)}{k^2} = \left(\frac{1}{k}\right)^2 \ln\left(\frac{1}{k}\right) \longrightarrow 0 \quad ???$$

Per $t \to 0^+$: $\quad t^2 \ln(t) \longrightarrow 0 \quad$ (limite notevole)

Alternativa: $\quad \dfrac{\ln\left(\frac{1}{k}\right)}{k^2} = -\dfrac{\ln(k)}{k^2} \longrightarrow 0 \quad$ gerarchia infiniti

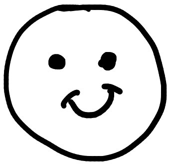
Provo con:

$$x_k = k, \qquad y_k = \underbrace{-\frac{1}{k}}_{\to -\frac{1}{k}} + \frac{e^{-k^2}}{k} = \frac{\overbrace{e^{-k^2} - 1}^{<0}}{k}$$

Valuto:

$$f(x_k, y_k) = \frac{\ln\left(1 - 1 + e^{-k^2}\right)}{k^2 + \left(-\frac{1}{k} + \frac{e^{-k}}{k}\right)^2} = \frac{-k^2}{k^2 + \underbrace{(\ldots)^2}_{\to 0}}$$

$$\sim \frac{-k^2}{k^2} \longrightarrow -1 \neq 0$$



Conclusione: $\nexists \displaystyle\lim_{\|(x,y)\| \to +\infty} f(x,y)$. $\square$

Esempi sulla continuità

• $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\mathrm{dom}(f) = \mathbb{R}^2$

In $\mathbb{R}^2 \setminus \{(0,0)\}$: f è razionale

$\Rightarrow$ è continua in ogni punto

Verifico se f è continua in $(0,0)$; cioè se

$$\exists \lim_{(x,y)\to(0,0)} f(x,y) = \underbrace{f(0,0)}_{=0}$$

Oss: $\forall (x,y) \neq (0,0)$

$$\underset{0}{0} \leq |f(x,y)| = \left| \frac{xy(x^2-y^2)}{x^2+y^2} \right| = \underbrace{\frac{|xy|}{x^2+y^2}}_{\leq \frac{1}{2}} |x^2-y^2|$$

$$\leq \frac{1}{2} \underbrace{|x^2-y^2|}_{} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\overset{\text{TCO}}{\Longrightarrow} \quad \exists \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Quindi: f è continua in $(0,0)$, pertanto è continua in $\mathbb{R}^2$. $\square$

Suggerimento: in alternativa

$$g(\rho,\theta) = f(\rho\cos\theta, \rho\sin\theta)$$

$$= \frac{\rho\cos\theta\,\rho\sin\theta\,(\rho^2\cos^2\theta - \rho^2\sin^2\theta)}{\rho^2\cos^2\theta + \rho^2\sin^2\theta}$$

$$= \rho^2 \underbrace{\cos\theta\sin\theta\,(\cos^2\theta - \sin^2\theta)}_{\text{limitata in } [0,2\pi]}$$

$$\Rightarrow \quad |g(\rho,\theta)| \le \underbrace{\rho^2 \cdot M}_{\to 0} \qquad \square$$

- Stabilire se

$$f(x,y) = \begin{cases} \dfrac{1-\sqrt{1-x^2-y^2}}{x^2+y^2} & (x,y) \neq (0,0) \\ \dfrac{1}{2} & (x,y) = (0,0) \end{cases}$$

ammette estremi globali nel proprio dominio.

$$\mathrm{dom}(f) = \{(x,y) \mid 1-x^2-y^2 \geq 0\}$$

$$= \{(x,y) \mid x^2+y^2 \le 1\} = \underbrace{\overline{B}_1(0,0)}_{\text{chiuso e limitato}}$$

Oss: per il teor. di Weierstrass, basta stabilire se f è continua in $\overline{B}_1(0,0)$.

In $\overline{B}_1(0,0) \setminus \{(0,0)\}$ f è continua (composta ..)

Verifico se f è continua in $(0,0)$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\overbrace{1-\sqrt{1-(x^2+y^2)}}^{\to 0}}{\underbrace{x^2+y^2}_{\to 0}}$$

$t = x^2+y^2$

$$= \lim_{t\to 0^+} \frac{1-\sqrt{1-t}}{t} = (*)$$

$t \to 0: \quad \sqrt{1-t} = \;??$

$g(t) = \sqrt{1-t} \qquad g(0) = 1$

$g'(t) = \frac{1}{2\sqrt{1-t}}(-1) \qquad g'(0) = -\frac{1}{2}$

$\Rightarrow g(t) = 1 - \frac{1}{2}t + o(t)$

$$(*) = \lim_{t\to 0^+} \frac{1-\left(1-\frac{1}{2}t+o(t)\right)}{t} = \lim_{t\to 0^+} \frac{\frac{1}{2}t+o(t)}{t}$$

$$= \frac{1}{2} = f(0,0)$$

$\Rightarrow f$ è continua anche in $(0,0)$. $\square$

Esempi sulle derivate direzionali:

- $f(x_1,x_2) = \frac{x_1^2}{2} + x_2^2 \qquad \bar{x} = (3,1) \qquad \mathrm{dom}(f) = \mathbb{R}^2$

$v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\lim_{t\to 0} \frac{f(\bar{x}+tv) - f(\bar{x})}{t} =$$

$$\lim_{t\to 0} \frac{f\left((3,1)+t\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right) - f(3,1)}{t} =$$

$$\lim_{t\to 0} \frac{f\left(3+\frac{t}{\sqrt{2}},\ 1+\frac{t}{\sqrt{2}}\right) - f(3,1)}{t} =$$

$$\lim_{t\to 0} \frac{\frac{1}{2}\left(3+\frac{t}{\sqrt{2}}\right)^2 + \left(1+\frac{t}{\sqrt{2}}\right)^2 - \left(\frac{3^2}{2}+1^2\right)}{t} =$$

$$\lim_{t\to 0} \frac{\cancel{\frac{9}{2}} + \frac{3}{\sqrt{2}}t + \frac{t^2}{4} + \cancel{1} + \frac{2}{\sqrt{2}}t + \frac{t^2}{2} - \cancel{\frac{9}{2}} - \cancel{1}}{t} =$$

$$\lim_{t\to 0} \frac{\frac{5}{\sqrt{2}}t + \frac{3}{4}t^2}{t} = \lim_{t\to 0} \frac{5}{\sqrt{2}} + \frac{3}{4}t = \underbrace{\frac{5}{\sqrt{2}}}_{\in \mathbb{R}}$$

$\Rightarrow$ f è derivabile in $(3,1)$ nella direzione $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ e $\frac{\partial f}{\partial v}(3,1) = \frac{5}{\sqrt{2}}$.

- $f(x_1,x_2) = \frac{x_1^2}{2} + x_2^2 \qquad \bar{x} = \left(\frac{1}{2},1\right) \qquad v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$\lim_{t\to 0} \frac{f\left(\left(\frac{1}{2},1\right) + t\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right) - f\left(\frac{1}{2},1\right)}{t} =$$

$$\lim_{t\to 0} \frac{f\left(\frac{1}{2}+\frac{t}{\sqrt{2}},\ 1+\frac{t}{\sqrt{2}}\right) - f\left(\frac{1}{2},1\right)}{t} =$$

$$\lim_{t\to 0} \frac{\frac{1}{2}\left(\cancel{\frac{1}{4}} + \frac{t}{\sqrt{2}} + \frac{t^2}{2}\right) + \cancel{1} + \frac{2}{\sqrt{2}}t + \frac{t^2}{2} - \cancel{\frac{1}{2}\cdot\frac{1}{4}} - \cancel{1}}{t} =$$

$$\lim_{t\to 0} \frac{\left(\frac{1}{2\sqrt{2}} + \frac{2}{\sqrt{2}}\right)t + \left(\frac{1}{4}+\frac{1}{2}\right)t^2}{t} = \underbrace{\frac{5}{2\sqrt{2}}}_{\in \mathbb{R}} =: \frac{\partial f}{\partial v}\left(\frac{1}{2},1\right)$$

- $f(x_1, x_2) = |x_1 - 1| \, (x_1 + x_2)$ $\qquad \bar{x} = (1,3)$

$\text{dom}(f) = \mathbb{R}^2$ $\qquad \bar{v} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$\forall t \neq 0$:

$$\frac{f\left((1,3) + t\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\right) - f(1,3)}{t} =$$

$$\frac{f\left(1 + \frac{2}{\sqrt{5}}t, \; 3 + \frac{t}{\sqrt{5}}\right) - f(1,3)}{t} =$$

$$\frac{\left|1 + \frac{2}{\sqrt{5}}t - 1\right| \left(1 + \frac{2}{\sqrt{5}}t + 3 + \frac{t}{\sqrt{5}}\right) - 0}{t} =$$

$$\frac{\frac{2}{\sqrt{5}}|t|\left(4 + \frac{3}{\sqrt{5}}t\right)}{t} = \underbrace{\frac{2}{\sqrt{5}}}_{\text{cost.}} \; \underbrace{\frac{|t|}{t}}_{\text{non ha limite!}} \; \underbrace{\left(4 + \frac{3}{\sqrt{5}}t\right)}_{\to 4}$$

$t \to 0$:

f NON è derivabile in $(1,3)$ nella direzione $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

- $f(x_1, x_2) = |x_1 - 1| \, (x_1 + x_2)$ $\qquad \bar{x} = (1,3)$

$\qquad v = (0,1)$

$\forall t \neq 0$:

$$\frac{f((1,3) + t(0,1)) - f(1,3)}{t} = \frac{f(1, 3+t) - f(1,3)}{t}$$

$$= \frac{0 - 0}{t} = 0$$

$$\Rightarrow \exists \lim_{t \to 0} \frac{f((1,3)+t(0,1)) - f(1,3)}{t} = 0$$

$$\Rightarrow \exists \frac{\partial f}{\partial v}(1,3) = 0 .$$

Riesamino gli esempi:

- $f(x_1, x_2) = \frac{x_1^2}{2} + x_2^2 \qquad \bar{x} = (3,1) \qquad v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$g: \mathbb{R} \to \mathbb{R}$ t.c. $g(t) = f\left((3,1) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) =$

$$= f\left(3 + \frac{t}{\sqrt{2}}, 1 + \frac{t}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left(3 + \frac{t}{\sqrt{2}}\right)^2 + \left(1 + \frac{t}{\sqrt{2}}\right)^2$$

Oss: g è polinomiale $\Rightarrow$ g è derivabile in $\mathbb{R}$
$\Rightarrow$ g è derivabile in $\underline{t=0}$ ✓

$\forall t$: $\quad g'(t) = \frac{1}{2} \cdot 2\left(3 + \frac{t}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} + 2\left(1 + \frac{t}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}}$

$$\Rightarrow g'(0) = 3 \cdot \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}} \qquad ✓$$

- $f(x_1, x_2) = |x_1 - 1|(x_1 + x_2) \qquad \bar{x} = (1,3)$
$\qquad v = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$g: \mathbb{R} \to \mathbb{R}$ t.c. $\forall t \in \mathbb{R}$:

$g(t) = f\left(1 + t \cdot \frac{2}{\sqrt{5}}, 3 + t \cdot \frac{1}{\sqrt{5}}\right)$

$$= \left| 1 + \frac{2t}{\sqrt{5}} - 1 \right| \left(1 + \frac{2t}{\sqrt{5}} + 3 + \frac{t}{\sqrt{5}} \right)$$

$$= \frac{2}{\sqrt{5}} \, \underline{|t|} \left(4 + \frac{3}{\sqrt{5}} t \right)$$

non è derivabile in $t = 0$!!

Cambio $v = (0, 1)$

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(t) = f(1 + t \cdot 0,\ 3 + t \cdot 1)$

$$= f(1, 3 + t) = 0$$

$$\Rightarrow \exists\, g'(0) = 0 .$$