11 aprile 2024

Esempi

- Riprendo $f(x,y) = \frac{xy}{x^2+y^2}$, $\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

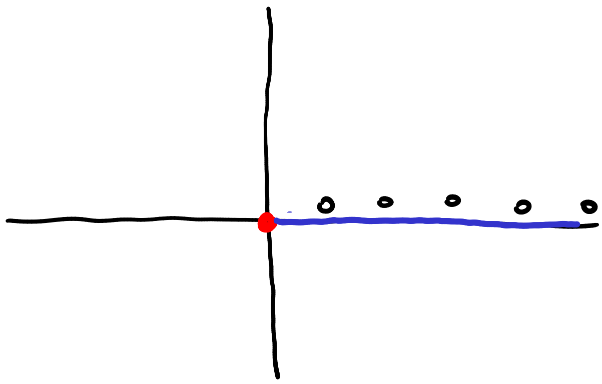
f razionale $\Rightarrow$ f continua

$\Rightarrow$ limiti significativi : $(x,y) \to (0,0)$

$\|(x,y)\| \to +\infty$

$\underset{\smile}{B} := \{(x,y) \mid x>0,\ y=0\} \subset \mathrm{dom}(f)$

- $(0,0) \in \mathrm{Dr}(B)$
- B illimitato



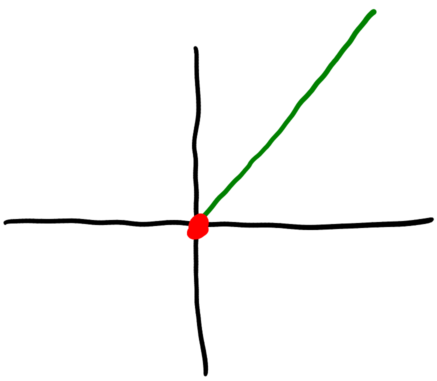
$\forall (x,y) \in B: \quad f_{|B}(x,y) = 0$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f_{|B}(x,y) = \lim_{x\to 0} f(x,0) = 0$$

$$\lim_{\|(x,y)\|\to+\infty} f_{|B}(x,y) = \lim_{x\to+\infty} f(x,0) = 0$$

$\Rightarrow$ <u>se</u> esistono, $\lim_{(x,y)\to(0,0)} f(x,y)$ e $\lim_{\|(x,y)\|\to+\infty} f(x,y)$ devono essere uguali a 0.

$C := \{(x,x) \mid x>0\} \subset \mathrm{dom}(f)$



$(0,0) \in \mathrm{Dr}(C)$, C illimitato

$$\lim_{(x,y)\to(0,0)} f_{|C}(x,y) = \lim_{x\to 0^+} f(x,x) = \lim_{x\to 0^+} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\lim_{\|(x,y)\|\to+\infty} f(x,y) = \lim_{x\to+\infty} f(x,x) = \lim_{x\to+\infty} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0$$

$$\Rightarrow \nexists \lim_{\|(x,y)\|\to+\infty} f(x,y). \qquad \square$$

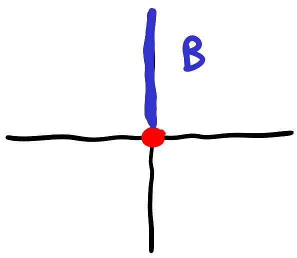
- $f(x,y) = \arctan\left(\dfrac{x}{x^2+y^2}\right)$

$\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

f continua (composta di razionale e arctan)

$\Rightarrow$ limiti significativi: $(x,y)\to(0,0)$, $\|(x,y)\|\to+\infty$

$B := \{(0,y) \mid y>0\} \subset \mathrm{dom}(f)$



$(0,0) \in D(B)$, B illimitato

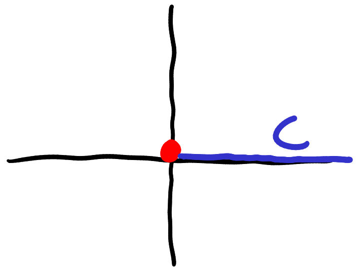
$$\lim_{(x,y)\to(0,0)} f_{|B}(x,y) = \lim_{y\to 0^+} \underbrace{f(0,y)}_{\equiv 0} = 0$$

(se esiste, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$)

$$\lim_{\|(x,y)\|\to+\infty} f_{|B}(x,y) = \lim_{y\to+\infty} f(0,y) = 0$$

(. . .)

Considero $C := \{(x,0) \mid x > 0\}$



$$\lim_{(x,y)\to(0,0)} f|_C (x,y) = \lim_{x\to 0^+} f(x,0)$$

$$= \lim_{x\to 0^+} \arctan\left(\frac{x}{x^2+0^2}\right) = \lim_{x\to 0^+} \arctan\Big(\underbrace{\frac{1}{x}}_{\to +\infty}\Big) = \frac{\pi}{2} \neq 0$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

$$\lim_{\|(x,y)\|\to+\infty} f|_C (x,y) = \lim_{x\to+\infty} \arctan\Big(\underbrace{\frac{1}{x}}_{\to 0}\Big) = 0 \qquad ???$$

Dopo ampia riflessione (!) congetturo che

$$\exists \lim_{\|(x,y)\|\to+\infty} f(x,y) = 0$$

Dimostro la congettura attraverso la definizione di limite.

Prendo $(x_k, y_k) \subset \mathbb{R}^2 \setminus \{(0,0)\}$ t.c. $x_k^2 + y_k^2 \to +\infty$

Valuto

$$0 \le \left|\frac{x_k}{x_k^2+y_k^2}\right| = \underbrace{\frac{|x_k|}{\sqrt{x_k^2+y_k^2}}}_{\le 1} \cdot \underbrace{\frac{1}{\sqrt{x_k^2+y_k^2}}}_{\ge 0} \le \frac{1}{\sqrt{x_k^2+y_k^2}}$$

$$\sqrt{a^2+b^2} \ge \sqrt{a^2} = |a| \Rightarrow \frac{|a|}{\sqrt{a^2+b^2}} \le 1$$

Quindi, $\forall k$:

$$\underbrace{0}_{0} \le \left|\frac{x_k}{x_k^2+y_k^2}\right| \le \underbrace{\frac{1}{\underbrace{\sqrt{x_k^2+y_k^2}}_{\to+\infty}}}_{\to 0}$$

TCO

$$\Rightarrow \quad \frac{x_k}{x_k^2+y_k^2} \to 0$$

continuità di arctan $\Rightarrow$

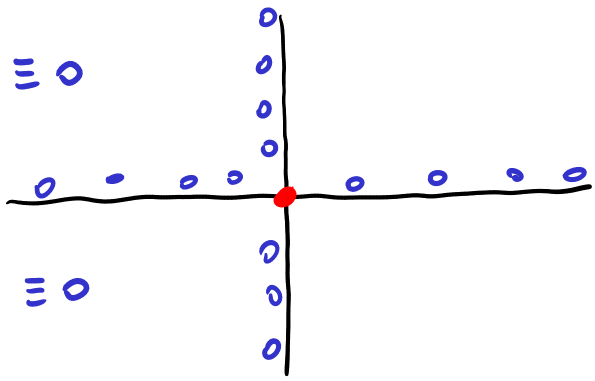
$$\underbrace{\arctan\left(\frac{x_k}{x_k^2+y_k^2}\right)}_{= f(x_k, y_k)} \to 0$$

Conclusione : $\exists \lim\limits_{\|(x,y)\| \to +\infty} f(x,y) = 0.$ $\square$

- $f(x,y) = \dfrac{x^3 y^2}{4x^2+y^2}$

$\left.\begin{array}{l} \mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\} \\ f \text{ continua} \end{array}\right\} \Rightarrow$ lim. significativi: $(x,y) \to (0,0)$, $\|(x,y)\| \to +\infty$

$f|_{\{x=0,\, y\neq 0\}} \equiv 0$



$f|_{\{x\neq 0,\, y=0\}} \equiv 0$

Se esistono, i due limiti sono uguali a 0

Considero $C = \{(x,x) \mid x \neq 0\}$

$$f|_C(x,y) = f(x,x) = \frac{x^3 x^2}{4x^2+x^2} = \frac{x^3}{5}$$

$$\lim_{(x,y)\to(0,0)} f|_C(x,y) = \lim_{x\to 0} \frac{x^3}{5} = 0 \quad ??$$

$$\lim_{\|(x,y)\|\to+\infty} f|_C(x,y) = \lim_{\substack{|x|\to+\infty \\ x \to \pm\infty}} \frac{x^3}{5} = \pm\infty \neq 0$$

$$\Rightarrow \nexists \lim_{\|(x,y)\| \to +\infty} f(x,y)$$

Resta da verificare se $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Provo con la definizione:

prendo $(x_k, y_k) \to (0,0)$ e valuto

$$\underset{\underset{0}{\downarrow}}{0} \le |f(x_k,y_k)| = \frac{|x_k^3 y_k^2|}{4x_k^2 + y_k^2} = \underbrace{|x_k^3|}_{\ge 0} \underbrace{\frac{y_k^2}{4x_k^2 + y_k^2}}_{\le 1}$$

$$\le \underbrace{|x_k^3|}_{\to 0}$$

$$\overset{TCO}{\Rightarrow} f(x_k, y_k) \to 0.$$

Quindi: $\exists \displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. $\square$

Riprendo l'ultimo esempio utilizzando coordinate polari di centro $(0,0)$:

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta$$

$$\Rightarrow g(\rho,\theta) := f(\rho\cos\theta, \rho\sin\theta)$$

$$= \frac{\rho^3\cos^3\theta \, \rho^2\sin^2\theta}{4\rho^2\cos^2\theta + \rho^2\sin^2\theta}$$

$$= \frac{\rho^5}{\rho^2} \frac{\cos^3\theta \sin^2\theta}{4\cos^2\theta + \sin^2\theta}$$

$$= \rho^3 \cdot \frac{\cos^3\theta \sin^2\theta}{4\cos^2\theta + \underbrace{\sin^2\theta}_{\ne 0 \quad \forall\theta\in[0,2\pi]}}$$

funzione continua in $[0, 2\pi]$

Per il teor. di Weierstrass: $\exists M > 0$ t.c.

$$\forall \theta \in [0, 2\pi] : \quad \left| \frac{\cos^3\theta \sin^2\theta}{4\cos^2\theta + \sin^2\theta} \right| \le M$$

candidato limite

$$\Rightarrow \forall \rho > 0 : \quad |g(\rho, \theta) - 0| \le \rho^3 \cdot M \qquad \forall \theta \in [0, 2\pi]$$

$$\Rightarrow \sup_{\theta \in [0, 2\pi]} |g(\rho, \theta) - 0| \le \underbrace{\rho^3 \cdot M}_{\xrightarrow{\rho \to 0} 0}$$

$$\Rightarrow \lim_{\rho \to 0} \sup_{\theta \in [0, 2\pi]} |g(\rho, \theta) - 0| = 0$$

$$\Rightarrow \lim_{(x,y) \to (0,0)} |f(x,y) - 0| = 0 \qquad \square$$

Oss: riprendo $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad (\bar{x}, \bar{y}) = (0,0)$

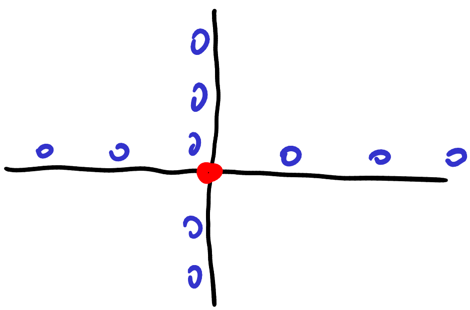
$$g(\rho, \theta) = f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^2 \cos\theta \sin\theta}{\rho^2}$$

$$= \cos\theta \sin\theta$$

Oss: $f(x, mx) = \dfrac{x \, mx}{x^2 + m^2 x^2} = \dfrac{m}{1 + m^2}$ $\quad \square$

- $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$ $\qquad \mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

  lim. sign: $(0,0)$, ∞



Provo con $\{(x,y) \mid y = x,\ x \neq 0\} =: B$

$$f_{|B}(x,y) = \frac{x^2 x}{x^4 + x^2} = \frac{x^3}{x^4 + x^2} = \frac{x}{x^2+1}$$

$$\lim_{(x,y)\to(0,0)} f_{|B}(x,y) = \lim_{x\to 0} \frac{x}{x^2+1} = 0 \quad ??$$

$$\lim_{\|(x,y)\|\to+\infty} f_{|B}(x,y) = \lim_{|x|\to+\infty} \frac{x}{x^2+1} = 0 \quad ??$$

$(m \in \mathbb{R}^*)$

Provo con $\{(x,y) \mid y = mx,\ x \neq 0\} =: B_m$

$$f_{|B_m}(x,y) = \frac{x^2\, mx}{x^4 + m^2 x^2} = \frac{mx}{x^2 + m^2} \xrightarrow[\substack{x\to 0 \\ x\to\pm\infty}]{} 0 \quad ???$$

Provo con $C_a := \{(x,y) \mid y = ax^2,\ x \neq 0\}$ $(a \in \mathbb{R}^*)$

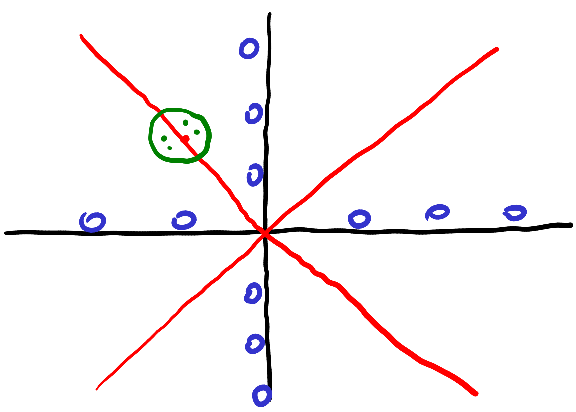
$$f_{|C_a}(x,y) = \frac{x^2\, ax^2}{x^4 + a^2 x^4} = \frac{a}{1+a^2}$$

$\uparrow$ funzione costante, con valore che dipende dalla parabola scelta

$\Rightarrow \nexists$ limiti per $(x,y)\to(0,0)$, $\|(x,y)\| \to +\infty$ $\square$

- $f(x,y) = \dfrac{x^2 y}{x^2 - y^2}$

$x^2 - y^2 \neq 0 \qquad y \neq \pm x$



$\mathrm{dom}(f) = \{(x,y) \mid x^2 - y^2 \neq 0\}$

(aperto, illimitato)

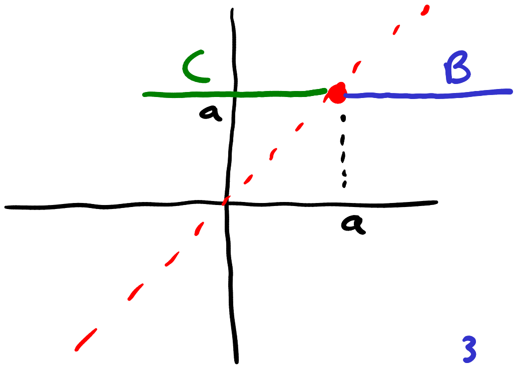
f razionale $\Rightarrow$ continua nel suo dominio

Limiti significativi:

$\|(x,y)\| \to +\infty$

$(x,y) \to (a,a)$
$(x,y) \to (a,-a)$ $\Big\}$ $a \neq 0$

$(x,y) \to (0,0)$



Inizio con (a,a), $a \neq 0$

$B := \{ (x,a) \mid x > a \}$

$C := \{ (x,a) \mid x < a \}$

$$\lim_{(x,y)\to(a,a)} f_{|_B}(x,y) = \lim_{x\to a^+} f(x,a) = \lim_{x\to a^+} \frac{\overbrace{x^2 a}^{\to a^3}}{\underbrace{x^2 - a^2}_{\to 0^+}}$$

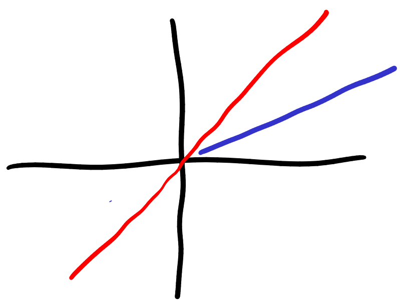
$$= \begin{cases} +\infty & a > 0 \\ -\infty & a < 0 \end{cases}$$

$$\lim_{(x,y)\to(a,a)} f_{|_C}(x,y) = \lim_{x\to a^-} \frac{\overbrace{x^2 a}^{\to a^3}}{\underbrace{x^2 - a^2}_{\to 0^-}} = \begin{cases} -\infty & a > 0 \\ +\infty & a < 0 \end{cases}$$

Quindi: $\nexists \lim_{(x,y)\to(a,a)} f(x,y)$.

Per $(a,-a)$, $a \neq 0$: stesso procedimento, stessa conclusione.

Per $\|(x,y)\| \to +\infty$:



$D := \{ (x, \frac{x}{2}) \mid x > 0 \}$

$$\lim_{\|(x,y)\|\to+\infty} f_{|_D}(x,y) = \lim_{x\to+\infty} f(x, \tfrac{x}{2})$$

$$= \lim_{x \to +\infty} \frac{x^2 \frac{x}{2}}{x^2 - \frac{x^2}{4}} = \lim_{x \to +\infty} \frac{\frac{x}{2}}{\frac{3}{4}} = +\infty \neq 0$$

$\Rightarrow \nexists \lim$ per $\|(x,y)\| \to +\infty$.

Da completare: $(x,y) \to (0,0)$.