Confronto tra la condizione

(*) presi due elementi distinti (u_1, v_1), $(u_2, v_2) \in K$, di cui uno almeno interno a K, si ha

$$\sigma(u_1, v_1) \neq \sigma(u_2, v_2)$$

e la condizione

(•) presi due elementi distinti (u_1, v_1), $(u_2, v_2) \in K$, entrambi interni a K, si ha

$$\sigma(u_1, v_1) \neq \sigma(u_2, v_2)$$

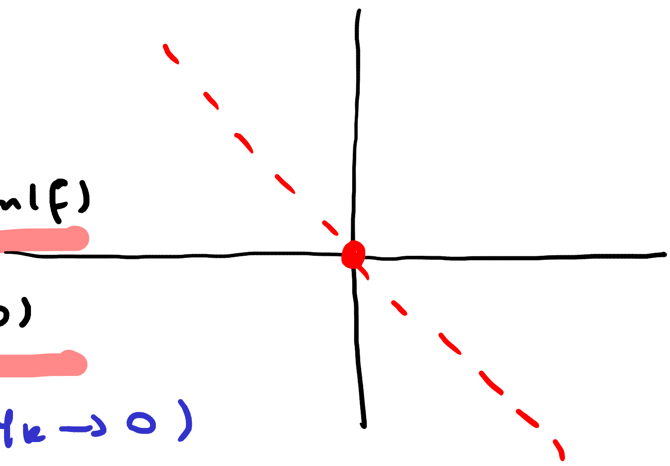
(cioè: la restrizione di σ all'interno di K è ingettiva)

Esempi

- Verifico che $\displaystyle\lim_{(x,y)\to(0,0)} \underbrace{(x^2+y^2)\sin\left(\frac{1}{x+y}\right)}_{=:\,f(x,y)} = 0$

$$\mathrm{dom}(f) = \{(x,y) \in \mathbb{R}^2 \mid x+y \neq 0\}$$

$(0,0) \in \mathrm{Dr}(\mathrm{dom}(f))$ ✓



Prendo $((x_k, y_k)) \subset \mathrm{dom}(f)$

t.c. $(x_k, y_k) \to (0,0)$

(cioè: $x_k \to 0$, $y_k \to 0$)

Valuto

$$f(x_k,y_k) = \underbrace{\underbrace{(x_k^2+y_k^2)}_{\to 0}}_{\text{infinitesima}} \cdot \underbrace{\sin\left(\frac{1}{\underbrace{x_k+y_k}_{\to 0}}\right)}_{\text{limitata}}$$

$? \to \pm\infty$

corall. TCO $\Rightarrow$ $f(x_k,y_k) \to 0$

In alternativa:

$$\forall k: \quad \underset{\to 0}{0} \le |f(x_k,y_k)| = \underbrace{(x_k^2+y_k^2)}_{\ge 0} \underbrace{\left|\sin\left(\frac{1}{x_k+y_k}\right)\right|}_{\le 1}$$

$$\le \underbrace{x_k^2+y_k^2}_{\to 0}$$

$\overset{TCO}{\Rightarrow}$ $|f(x_k,y_k)| \to 0 \iff f(x_k,y_k) \to 0$ $\square$

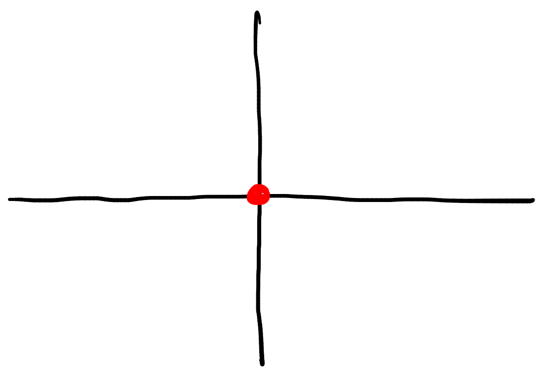
- $$\lim_{(x,y)\to(0,0)} \underbrace{\frac{3x^3+2x^2+2y^2}{x^2+y^2}}_{=: f(x,y)} = 2$$

$\mathrm{dom}(f) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$(0,0) \in Dr(\mathrm{dom}(f))$

Prendo $((x_k,y_k)) \subset \mathrm{dom}(f)$ t.c.

$(x_k,y_k) \to (0,0)$



Valuto

$$f(x_k, y_k) = \frac{3x_k^3 + 2x_k^2 + 2y_k^2}{x_k^2 + y_k^2}$$

$$= \frac{3x_k^3}{x_k^2+y_k^2} + 2$$

Oss: $f(x_k, y_k) \to 2 \iff$

$$\frac{3x_k^3}{x_k^2+y_k^2} \to 0$$

Oss:

$$\underset{\downarrow \atop 0}{0} \le \left| \frac{3x_k^3}{x_k^2+y_k^2} \right| = \underbrace{3|x_k|}_{\ge 0} \underbrace{\left(\frac{x_k^2}{x_k^2+y_k^2}\right)}_{\le 1} \le \underbrace{3|x_k| \cdot 1}_{\downarrow \atop 0}$$

$$\overset{TCO}{\Rightarrow} \left| \frac{3x_k^3}{x_k^2+y_k^2} \right| \to 0 \iff \frac{3x_k^3}{x_k^2+y_k^2} \to 0 \quad \square$$

- $$\lim_{(x,y)\to(1,1)} \overbrace{\frac{(y-1)^4}{\underbrace{x^2+y^2+2(1-x-y)}_{=:\, g(x,y)}}}^{=:\, f(x,y)} = 0$$

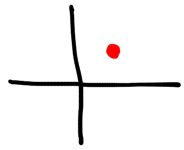
Oss: $g(x,y) = x^2 + y^2 + 1 + 1 - 2x - 2y$

$$= (x-1)^2 + (y-1)^2$$

$g(x,y) = 0 \iff (x-1)^2 + (y-1)^2 = 0$

$\iff x-1 = y-1 = 0 \iff x=1$ e $y=1$

$\Rightarrow \quad \operatorname{dom}(f) = \mathbb{R}^2 \setminus \{(1,1)\}$



$(1,1)$ punto di accumulazione per $\operatorname{dom}(f)$

Prendo $((x_k,y_k)) \subset \operatorname{dom}(f)$ t.c. $(x_k,y_k) \to (1,1)$

$(x_k \to 1, \; y_k \to 1)$

Valuto

$$f(x_k,y_k) = \frac{(y_k-1)^4}{(x_k-1)^2+(y_k-1)^2}$$

Oss:

$$\underset{\downarrow\, 0}{0} \le f(x_k,y_k) = \underbrace{(y_k-1)^2}_{\geq 0} \underbrace{\left(\frac{(y_k-1)^2}{(x_k-1)^2+(y_k-1)^2}\right)}_{\leq 1} \le \underset{\searrow\, 0}{(y_k-1)^2}$$

$$\overset{TCo}{\Rightarrow} \quad f(x_k,y_k) \longrightarrow 0 \quad \square$$

- $$\lim_{(x,y)\to(0,0)} \underbrace{\left(\frac{x^3 y}{x^4+y^2}\right)}_{=:f(x,y)} = 0$$

$$x^4+y^2=0 \iff x^4 = y^2 = 0 \iff x=y=0$$

$\Rightarrow \quad \operatorname{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

Prendo $((x_k,y_k)) \subset \operatorname{dom}(f)$ tc. $(x_k,y_k) \to (0,0)$

Valuto :

$$0 \le |f(x_k,y_k)| = \frac{|x_k^3 y_k|}{x_k^4+y_k^2} =: (*)$$

Oss: $\forall a, b \in \mathbb{R}$:

$$0 \le (|a| - |b|)^2 = |a|^2 + |b|^2 - 2|a||b|$$

$$\Rightarrow \quad 2|a||b| \le a^2 + b^2$$

$$\Leftrightarrow \quad |a||b| \le \frac{a^2+b^2}{2}$$

$(a,b) \neq (0,0)$

$$\Leftrightarrow \quad \frac{|a||b|}{a^2+b^2} \le \frac{1}{2}$$

$$(*) = |x_k| \frac{\overbrace{|x_k^2|}^{a} \overbrace{|y_k|}^{b}}{\underbrace{x_k^4}_{a^2} + \underbrace{y_k^2}_{b^2}} \le |x_k| \cdot \frac{1}{2}$$

Quindi:

$$\forall k \quad \underset{\downarrow\, 0}{0} \le |f(x_k, y_k)| \le \frac{1}{2} \underbrace{|x_k|}_{\to 0}$$

$$\overset{TCO}{\Longrightarrow} \quad f(x_k, y_k) \to 0.$$

- $$\lim_{(x,y,z) \to (0,0,0)} \underbrace{\left(\frac{(x+y)z^4}{x^4 + y^2 + z^4} \right)}_{=: f(x,y,z)} = 0$$

$$\mathrm{dom}(f) = \mathbb{R}^3 \setminus \{(0,0,0)\}$$

$(0,0,0) \in Dr(\mathrm{dom}(f))$

Prendo $((x_k, y_k, z_k)) \subset \mathrm{dom}(f)$ t.c.

$$(x_k, y_k, z_k) \to (0,0,0)$$

Valuto:

$$0 \le |f(x_k, y_k, z_k)| = \frac{|(x_k+y_k)\, z_k^4|}{x_k^4 + y_k^2 + z_k^4}$$
$$\downarrow$$
$$0$$

$$= |x_k + y_k| \underbrace{\frac{z_k^4}{x_k^4 + y_k^2 + z_k^4}}_{\le 1} \le \underbrace{|x_k + y_k|}_{\to |0 + 0| \to 0}$$

$$\overset{\text{TCO}}{\Longrightarrow} \quad f(x_k, y_k, z_k) \to 0 \qquad \square$$

Richiamo:

$A \subseteq \mathbb{R}^n$ limitato $\overset{def}{\iff}$ $\exists M > 0$ tc. $\forall x \in A$: $\|x\| \le M$

A illimitato $\iff$ $\forall M > 0 \ \exists x \in A$ tc. $\|x\| > M$

$\Rightarrow \forall k \in \mathbb{N} \ \exists x_k \in A$ tc. $\|x_k\| > k$
$$\downarrow$$
$$+\infty$$

$$\overset{\text{TDO}}{\Longrightarrow} \quad \|x_k\| \to +\infty \qquad \square$$

Oss. (cond. suff. affinché $\|x_k\| \to +\infty$)

Suppongo $|x_{k,i}| \to +\infty$

Ricordo: $|x_{k,i}| \le \|x_k\| \quad \forall k$

$$\overset{\text{TDO}}{\Longrightarrow} \quad \|x_k\| \to +\infty$$

Controesempio:

$$x_k = \begin{cases} k & k \text{ pari} \\ 0 & k \text{ dispari} \end{cases} \qquad y_k = \begin{cases} 0 & k \text{ pari} \\ k & k \text{ dispari} \end{cases}$$

$|x_k| = x_k$ non div. positivamente

$|y_k|$ " " "

Però: $(x_k, y_k) = \begin{cases} (k,0) & k \text{ pari} \\ (0,k) & k \text{ dispari} \end{cases}$

$$\Rightarrow \quad \|(x_k,y_k)\| = \begin{cases} \sqrt{k^2+0^2} & k \text{ pari} \\ \sqrt{0^2+k^2} & k \text{ disp.} \end{cases} = k \quad \forall k$$

$$\Rightarrow \quad \|(x_k,y_k)\| \to +\infty \qquad \square$$

Esempio

Verifico che $\displaystyle\lim_{\|(x,y)\|\to+\infty} \underbrace{\frac{x^2+y^2}{x^4+y^4}}_{=:f(x,y)} = 0$

$\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$ illimitato

Prendo $((x_k,y_k)) \subset \mathbb{R}^2 \setminus \{(0,0)\}$ t.c.

$\|(x_k,y_k)\| \to +\infty$ (cioè: $x_k^2+y_k^2 \to +\infty$)

Valuto

$$f(x_k,y_k) = \frac{x_k^2+y_k^2}{x_k^4+y_k^4}$$

Oss: $\forall a, b \in \mathbb{R}$:

$$a^4 + b^4 = a^4 + b^4 + 2a^2b^2 - 2a^2b^2$$

$$= (a^2+b^2)^2 - \underbrace{2a^2b^2}_{\leq a^4+b^4}$$

$$\geq (a^2+b^2)^2 - (a^4+b^4)$$

$$\Rightarrow \quad 2(a^4+b^4) \geq (a^2+b^2)^2$$

$$\Leftrightarrow \quad a^4+b^4 \geq \frac{(a^2+b^2)^2}{2}$$

$(a,b) \neq (0,0)$
$$\Leftrightarrow \quad \frac{1}{a^4+b^4} \leq \frac{2}{(a^2+b^2)^2}$$

$$\forall k: \quad \frac{1}{x_k^4+y_k^4} \leq \frac{2}{(x_k^2+y_k^2)^2}$$

$$\Rightarrow \quad \frac{x_k^2+y_k^2}{x_k^4+y_k^4} \leq 2\,\frac{(x_k^2+y_k^2)}{(x_k^2+y_k^2)^2} = \frac{2}{x_k^2+y_k^2}$$

$$\Rightarrow \quad \forall k: \quad \underset{\downarrow \atop 0}{0} \leq f(x_k, y_k) \leq \underset{\downarrow \atop 0}{\frac{2}{\underbrace{(x_k^2+y_k^2)}_{\to +\infty}}}$$

T CO
$$\Rightarrow \quad f(x_k, y_k) \to 0. \qquad \square$$

Esempi

- $$\lim_{(x,y)\to(0,0)} \underbrace{\frac{1}{x^2+y^4}}_{=:f(x,y)} = +\infty$$

$dom(f) = \mathbb{R}^2 \setminus \{(0,0)\}$, $(0,0)$ ✓

Prendo $((x_k,y_k)) \subset dom(f)$, $(x_k,y_k) \to (0,0)$

Valuto

$$f(x_k,y_k) = \frac{1}{\underbrace{\underbrace{x_k^2}_{\to 0^+} + \underbrace{y_k^4}_{\to 0^+}}_{\to 0^+}} \longrightarrow +\infty \qquad \square$$

- $$\lim_{\|(x,y)\|\to+\infty} \frac{x^4+y^4}{x^2+y^2} = +\infty$$

ha senso perché
$dom(f) = \mathbb{R}^2 \setminus \{(0,0)\}$
illimitato

[faccio finta di
non vedere che
...]

Prendo $(x_k,y_k) \subset \mathbb{R}^2 \setminus \{(0,0)\}$ t.c. $x_k^2 + y_k^2 \to +\infty$

Valuto

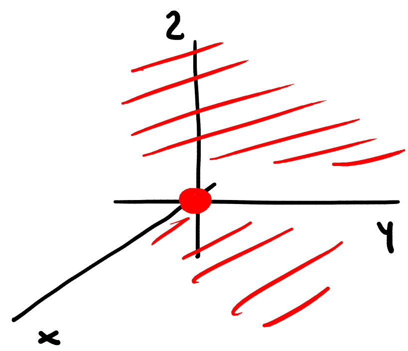
$$f(x_k,y_k) = \frac{x_k^4+y_k^4}{x_k^2+y_k^2} \geq \frac{(x_k^2+y_k^2)^{\cancel{2}}}{\underbrace{2\,(x_k^2+y_k^2)}_{\to +\infty}}$$

TDC $\Rightarrow$ $f(x_k,y_k) \to +\infty$. $\square$

• $\lim\limits_{(x,y,z)\to(0,0,0)} \left\| \left(\frac{1}{x^2}, 3+yz^3, x^2y \right) \right\| = +\infty$

$f(x,y,z) := \left(\frac{1}{x^2}, 3+yz^3, x^2y \right)$

$\mathrm{dom}(f) = \{ (x,y,z) \in \mathbb{R}^3 \mid x \neq 0 \}$



Prendo $((x_k, y_k, z_k)) \subset \mathrm{dom}(f)$

t.c. $(x_k, y_k, z_k) \to (0,0,0)$

Valuto:

$$|f_1(x_k, y_k, z_k)| = \left| \frac{1}{x_k^2} \right| = \frac{1}{\underbrace{x_k^2}_{\to 0^+}} \longrightarrow +\infty$$

$\Rightarrow \| f(x_k, y_k, z_k) \| \to +\infty \qquad \square$

$f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m, \quad \bar{x} \in A$

• f continua in $\bar{x} \overset{\text{def}}{\iff}$

$\Downarrow \Uparrow \quad \forall (x_k) \subset A$ t.c. $x_k \to \bar{x}$: $f(x_k) \to f(\bar{x})$

• $\lim\limits_{x \to \bar{x}} f(x) = f(\bar{x}) \overset{\text{def}}{\iff}$

$\forall (x_k) \subset A \setminus \{\bar{x}\}$ tc $x_k \to \bar{x}$: $f(x_k) \to f(\bar{x})$

Es: $f(x,y) = \dfrac{xy}{x^2+y^2}$

limiti significativi

$(x,y) \to (0,0)$ ✗

$\|(x,y)\| \to +\infty$ ✗

$dom(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

f continua