26 marzo 2024

Riprendo l'esempio.

$E_1 = \{ x \in \mathbb{R}^n \mid \|x - x_0\| < r \}$ aperto (già visto)

$E_2 = \{ \ldots\ldots \leq r \}$

$E_3 = \{ \ldots\ldots = r \}$

$E_4 = \{ \ldots\ldots > r \}$ aperto (visto)

$E_5 = \{ \ldots\ldots \geq r \}$

Oss: $E_2 = E_4^c$, E_4 aperto $\Rightarrow$ E_2 chiuso

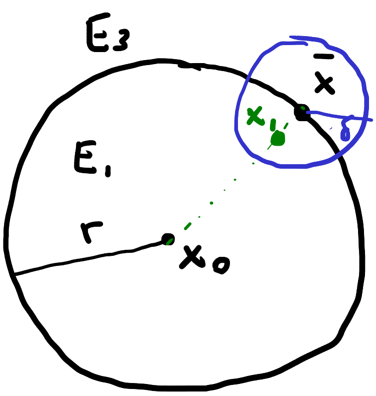
$E_5 = E_1^c$, E_1 aperto $\Rightarrow$ E_5 chiuso

$E_3 = E_2 \cap E_5$, E_2, E_5 chiusi $\Rightarrow$ E_3 chiuso

E_3 chiuso $\Rightarrow$ $\partial E_3 \subseteq E_3$

Verifico che $E_3 \subseteq \partial E_3$ (da cui: $\partial E_3 = E_3$)

Fisso $\bar{x} \in E_3$



Fisso $\delta > 0$; suppongo $\delta < r$ (non è restrittivo)

Considero:

$$x_1 := \underbrace{x_0}_{\in \mathbb{R}^n} + \underbrace{\left(1 - \frac{\delta}{2r}\right)}_{\in \mathbb{R}} \underbrace{(\bar{x} - x_0)}_{\in \mathbb{R}^n} \quad \in \mathbb{R}^n$$

$$x_2 := x_0 + \left(1 + \frac{\delta}{2r}\right)(\bar{x} - x_0) \quad \in \mathbb{R}^n$$

Calcolo:

$$\|x_1 - \bar{x}\| = \left\| x_0 + \left(1 - \frac{\delta}{2r}\right)(\bar{x} - x_0) - \bar{x} \right\|$$

$$= \left\| \left(1 - \frac{\delta}{2r}\right)(\bar{x} - x_0) - (\bar{x} - x_0) \right\|$$

$$= \left\| \left(1 - \frac{\delta}{2r} - 1\right)(\bar{x} - x_0) \right\|$$

$$= \left| -\frac{\delta}{2r} \right| \underbrace{\|\bar{x} - x_0\|}_{= r} = \frac{\delta}{2r} \cdot r = \frac{\delta}{2} < \delta$$

$$\Rightarrow x_1 \in B_\delta(\bar{x})$$

Calcolo

$$\|x_1 - x_0\| = \left\| x_0 + \left(1 - \frac{\delta}{2r}\right)(\bar{x} - x_0) - x_0 \right\|$$

$$= \underbrace{\left| 1 - \frac{\delta}{2r} \right|}_{= \underbrace{1 - \frac{\delta}{2r}}_{< 1}} \underbrace{\|\bar{x} - x_0\|}_{= r} < r$$

$$\Rightarrow x_1 \in E_1 \qquad (\Rightarrow x_1 \in E_3^c)$$

Dunque: $B_\delta(\bar{x})$ contiene $\bar{x}$, che è elemento di E_3, e x_1, che è elemento di E_3^c

Dato che δ è arbitrario, $\bar{x}$ è punto di frontiera per E_3.

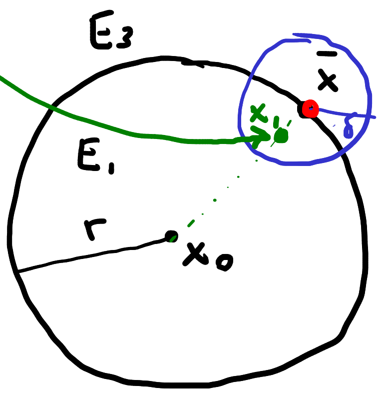
Qual è la frontiera di E_1?

Oss: $\quad \partial E_1 \subseteq E_3 \qquad$ (facile!)

Prendo $\bar{x} \in E_3$ e mostro che $\bar{x} \in \partial E_1$.

Prendo $\delta > 0$ (anche $\delta < r$)

$B_\delta(\bar{x})$ contiene un elem. di E_1 e un elem. di E_1^c ? sì!



Dunque: $\quad \partial E_1 = E_3$

$$(\Rightarrow \quad \partial E_5 = E_3)$$

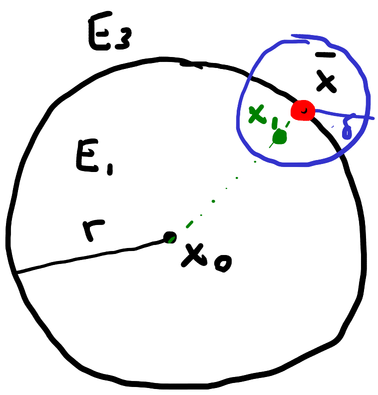
$E_5 = E_1^c$

Poi: $\quad \partial E_4 \subseteq E_3 \qquad$ (facile)

Viceversa: $\quad \bar{x} \in E_3 \overset{??}{\Longrightarrow} \bar{x} \in \partial E_4$

Preso $0 < \delta$ ($< r$): esistono elem. di E_4 e del suo complementare appartenenti a $B_\delta(\bar{x})$?

x_2



Per esercizio:
verificare che

$x_2 \in E_4 \qquad$ (cioè: $\|x_2 - x_0\| > r$)

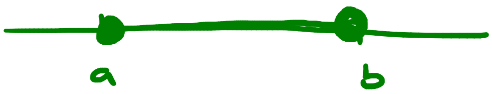
e $\quad x_2 \in B_\delta(\bar{x}) \qquad$ (cioè: $\|x_2 - \bar{x}\| < \delta$).

$\square$

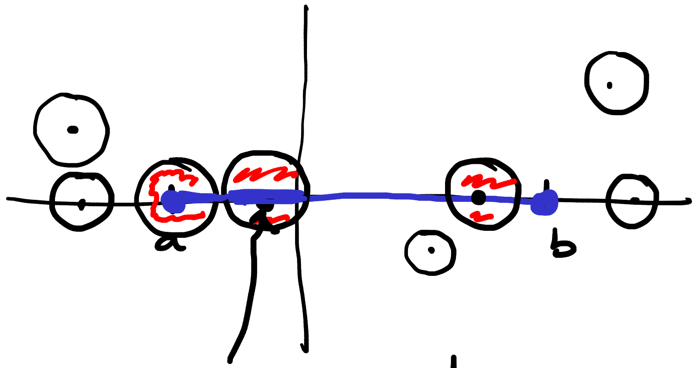
Es:

$E = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b],\ y = 0\}$

$\tilde{E} = [a,b] \quad (\subseteq \mathbb{R})$



$\partial \tilde{E} = \{a, b\}$



di frontiera!

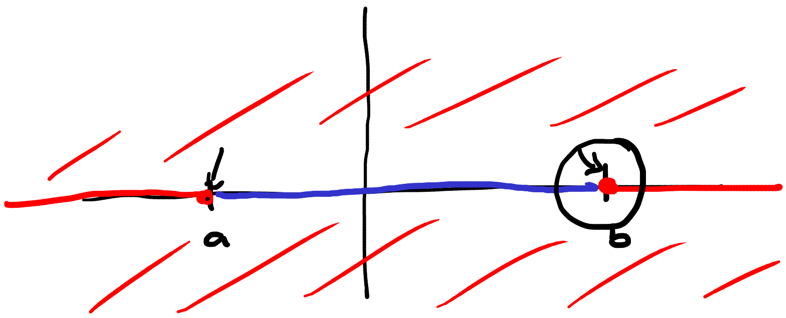
$\partial E = E \quad (\Rightarrow E$ chiuso$)$

Oss: $E = [a,b] \times \{0\}$

- $F = \{(x,y) \in \mathbb{R}^2 \mid x \in (a,b),\ y = 0\}$

$(= (a,b) \times \{0\})$

intervallo aperto



$\partial F = E$

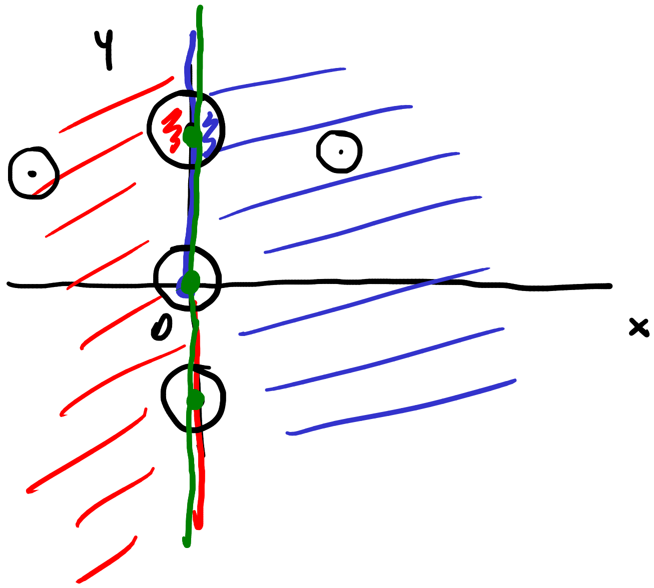
Oss: F non è chiuso $\left(\begin{matrix}(a,0)\\(b,0)\end{matrix} \in \partial F, \quad \begin{matrix}(a,0)\\(b,0)\end{matrix} \notin F \right)$

F non è aperto $\quad (F \cap \partial F \neq \emptyset)$

- $E = \{(x,y)\in\mathbb{R}^2 \mid x>0\} \cup \{(0,y)\in\mathbb{R}^2 \mid y\geq 0\}$

$\partial E = \{(0,y) \mid y\in\mathbb{R}\}$

$= \{0\}\times\mathbb{R}$



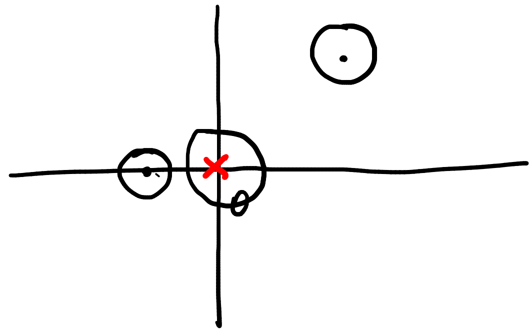
E aperto? No!

E chiuso? No!

- $E = \mathbb{R}^2 \setminus \{(0,0)\}$

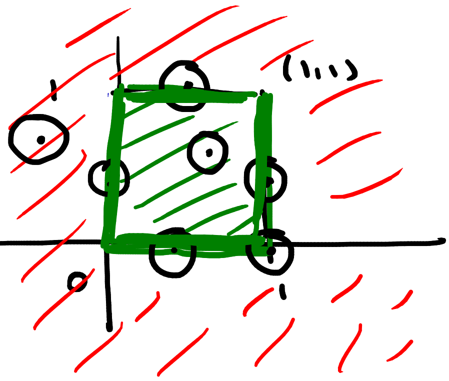
$\partial E = \{(0,0)\}$

E aperto



- $F = [0,1]\times[0,1]$

$= \{(x,y) \mid 0\leq x\leq 1,\ 0\leq y\leq 1\}$



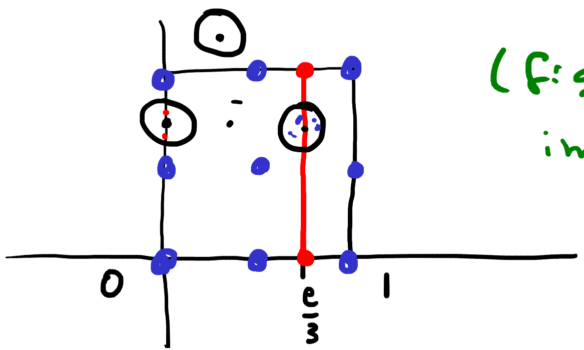
$\partial F = ([0,1]\times\{0\}) \cup (\{1\}\times[0,1]) \cup ([0,1]\times\{1\}) \cup (\{0\}\times[0,1])$

$E = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,1]\cap\mathbb{Q},\ y\in[0,1]\cap\mathbb{Q}\}$

$\partial E = [0,1]\times[0,1]$

E non chiuso

$\mathrm{int}(E) = \emptyset$



(figura impossibile)

E limitato $\iff \exists M \in \mathbb{R}_+^*$ t.c. $\|x\| \leq M \quad \forall x \in E$

$\|x\| = \|x-0\| \leq M \quad \forall x \in E$

palla chiusa

$\iff \forall x \in E : \quad x \in \overline{B_M}(0)$

$\iff \quad E \subseteq \overline{B_M}(0)$

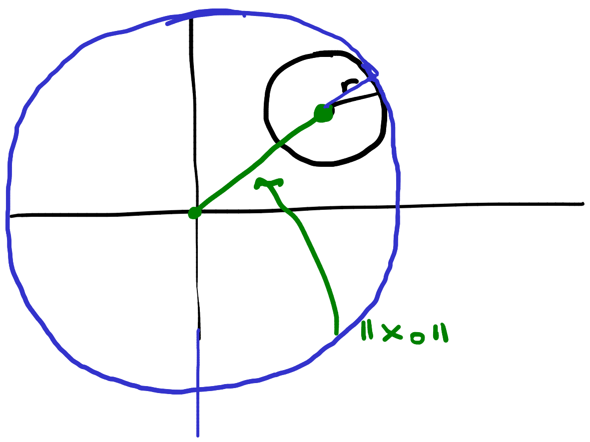
Oss: $\quad B_r(x_0)$ è limitata?

$x \in B_r(x_0) \iff \|x - x_0\| < r$

drs. triang.

$$\Rightarrow \quad \|x\| = \|x - x_0 + x_0\| \leq \|x - x_0\| + \|x_0\|$$

$$\leq r + \|x_0\| =: M$$



Oss: $\quad n=1 \quad x, y \in \mathbb{R}, \quad x < y$

Dico che

$$\{z \in \mathbb{R} \mid z = (1-t)x + ty, \ t \in [0,1]\} = [x,y]$$

Prendo $t \in [0,1]$ e $\quad z = (1-t)x + ty$

$$= x + t(y-x)$$

$$x<y \implies \left.\begin{matrix} y-x>0 \\ t\in[0,1] \end{matrix}\right\} \implies t(y-x)\geq 0$$

$$\implies x+\underbrace{t(y-x)}_{\geq 0} \geq x$$

$$\iff z\geq x$$

$$z = x+\underbrace{t}_{\leq 1}(\underbrace{y-x}_{>0}) \leq x+y-x = y$$

$$\implies z\in[x,y] \quad \checkmark$$

Prendo $z\in[x,y]$, cioè: $x\leq z\leq y$

Scelgo $t := \dfrac{z-x}{y-x}$ ⊙

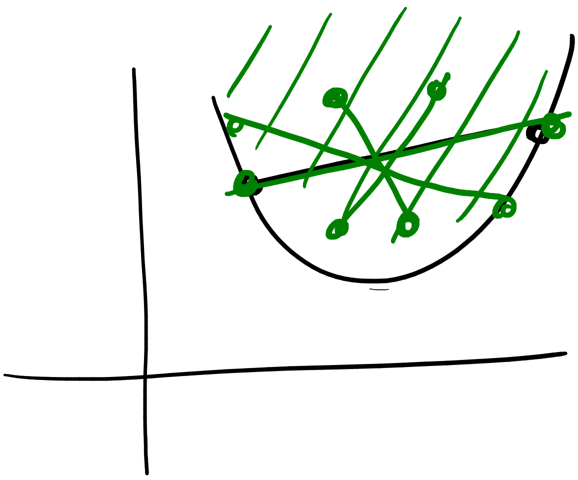
Osservo che: $t\geq 0$ $(z\geq x,\ y>x)$

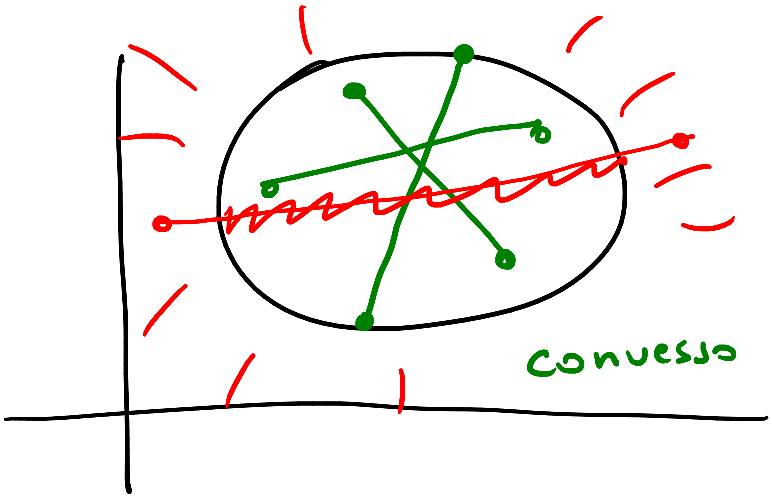
$t\leq 1$ $(z-x\leq y-x)$

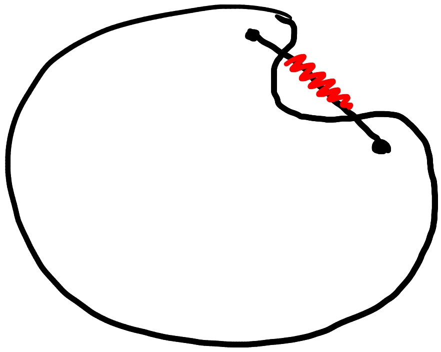
quindi: $t\in[0,1]$

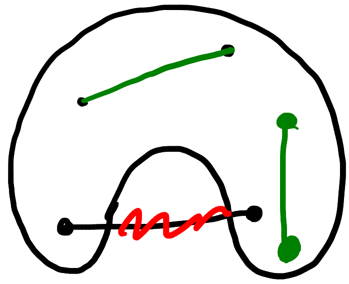
Da ⊙ : $(y-x)t = z-x$

$$\iff z = x+t(y-x)$$
$$= (1-t)x+ty \qquad \square$$

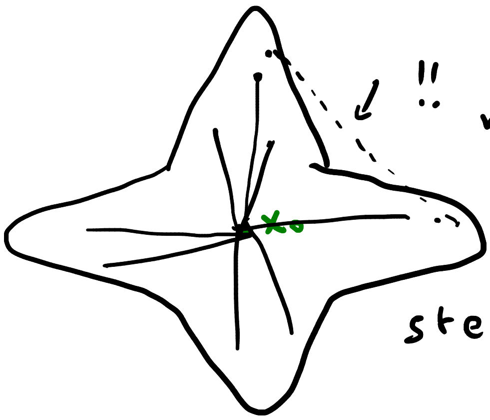






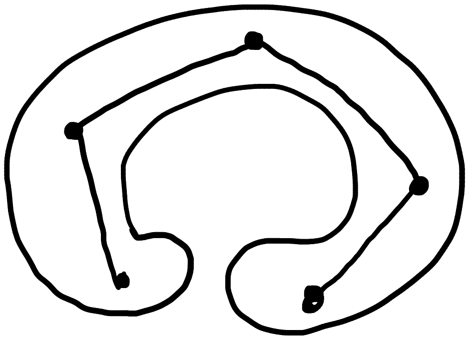


non convesso !



!! non convesso

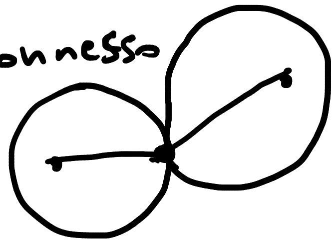
stellato

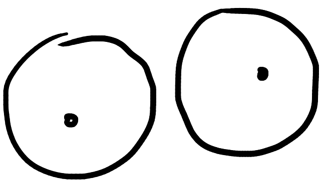


connesso per poligonali

(non stellato)

connesso





non connesso

Verifico che $B_r(x_0)$ è convesso.

Prendo $x, y \in B_r(x_0)$, $t \in (0,1)$ — $t=0$, $t=1$ niente da provare

Tesi: posto $z := (1-t)x + ty$, $z \in B_r(x_0)$

Calcolo:

$$\|z - x_0\| = \|(1-t)x + ty - \overset{\overset{(1-t)x_0 + tx_0}{\shortparallel}}{x_0}\|$$

$$= \|(1-t)x + ty - (1-t)x_0 - tx_0\|$$

$$= \| (1-t)(x-x_0) + t(y-x_0) \|$$

$$\leq \| (1-t)(x-x_0) \| + \| t(y-x_0) \|$$

$$= \underbrace{(1-t)}_{>0} \underbrace{\| x-x_0 \|}_{<r} + \underbrace{t}_{>0} \underbrace{\| y-x_0 \|}_{<r}$$

$$\underline{<} \; (1-t)r + tr = \underline{r} \qquad \square$$