21 marzo 2024

Esempi (sul metodo di Lagrange)

- $y'' - y = \dfrac{1}{1+e^{2t}}$ $\qquad t \in \mathbb{R}$

Determino un SFS dell'eq. omogenea associata

$P(\lambda) = \lambda^2 - 1$, radici $\lambda = 1$, $\lambda = -1$

$\Rightarrow$ SFS : $\varphi_1(t) = e^t$, $\varphi_2(t) = e^{-t}$

Determino una sol. particolare dell'equazione del tipo

$$\bar{\varphi} = \gamma_1 \varphi_1 + \gamma_2 \varphi_2$$

con γ_1 e γ_2 t.c.

$$\begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{1+e^{2t}} \end{pmatrix} \qquad \forall t \in \mathbb{R}$$

Risolvo :

$$\gamma_1'(t) = \frac{\begin{vmatrix} 0 & e^{-t} \\ \frac{1}{1+e^{2t}} & -e^{-t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}} = \frac{-\frac{e^{-t}}{1+e^{2t}}}{-1-1} = \frac{1}{2e^t(1+e^{2t})}$$

Calcolo:

$$\int \frac{1}{2e^t(1+e^{2t})}\,dt = \int \frac{1}{2e^{2t}(1+e^{2t})}\, e^t\,dt \qquad s = e^t$$

$$= \int \frac{1}{2s^2(1+s^2)}\,ds = \int \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{1+s^2}\right) ds$$

$$= -\frac{1}{2s} - \frac{1}{2}\arctan(s) + c = -\frac{e^{-t}}{2} - \frac{1}{2}\arctan(e^t) + c$$

Scelgo $\gamma_1(t) = -\frac{e^{-t}}{2} - \frac{1}{2}\arctan(e^t)$

$$\gamma_2'(t) = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{1}{1+e^{2t}} \end{vmatrix}}{-2} = -\frac{1}{2}\frac{e^t}{1+e^{2t}} \qquad \left[\begin{array}{c} s = e^t \\ \int -\frac{1}{2}\frac{1}{1+s^2}\,ds \end{array} \right]$$

Scelgo $\gamma_2(t) = -\frac{1}{2}\arctan(e^t)$

Dunque: una soluz. particolare dell' eq. è

$$\bar{\varphi}(t) = \gamma_1(t)\varphi_1(t) + \gamma_2(t)\varphi_2(t)$$

$$= -\left(\frac{e^{-t}}{2} + \frac{1}{2}\arctan(e^t)\right)e^t - \frac{1}{2}\arctan(e^t)\,e^{-t}$$

$$= -\frac{1}{2} - \frac{1}{2}\arctan(e^t)\,(e^t + e^{-t})$$

$$= -\frac{1}{2} - \arctan(e^t)\cdot\cosh(t)$$

L'integrale generale dell' eq. data è:

$$\left\{ c_1 e^t + c_2 e^{-t} - \frac{1}{2} - \arctan(e^t)\cosh(t),\ t\in\mathbb{R} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

- $y'' - 2y' + y = \frac{e^t}{1+t^2} \qquad t\in\mathbb{R}$

Determino un SFS dell' eq. omogenea associata:

$P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

$\Rightarrow$ SFS: $\varphi_1(t) = e^t, \qquad \varphi_2(t) = t\,e^t$

Determino una sol. particolare del tipo

$$\bar{\varphi} = \gamma_1 \varphi_1 + \gamma_2 \varphi_2$$

con γ_1 e γ_2 tali che

$$\begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix} \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^t}{1+t^2} \end{pmatrix} \qquad \forall t \in \mathbb{R}$$

Risolvo:

$$\gamma_1'(t) = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{1+t^2} & e^t + te^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{-\frac{t\,e^{2t}}{1+t^2}}{e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}}}$$

$$= -\frac{t}{1+t^2} = -\frac{1}{2}\,\frac{2t}{1+t^2}$$

Scelgo $\gamma_1(t) = -\frac{1}{2}\ln(1+t^2)$

$$\gamma_2'(t) = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{1+t^2} \end{vmatrix}}{e^{2t}} = \frac{\frac{e^{2t}}{1+t^2}}{e^{2t}} = \frac{1}{1+t^2}$$

Scelgo $\gamma_2(t) = \arctan(t)$

Quindi: una sol. particolare dell'eq. data è

$$\bar{\varphi}(t) = \gamma_1(t)\varphi_1(t) + \gamma_2(t)\varphi_2(t)$$

$$= -\frac{e^t}{2}\ln(1+t^2) + te^t \arctan(t)$$

Integrale generale: $c_1 e^t + c_2 te^t + \bar{\varphi}(t)$, $t \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R}$.

- $y''' + y' = \tan(t) \qquad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

SFS : $\qquad P(\lambda) = \lambda^3 + \lambda = \lambda(\lambda^2+1)$

radici $\quad \lambda = 0, \quad \lambda = i, \quad \lambda = -i$

$$\varphi_1(t) = e^{0t} = 1, \quad \varphi_2(t) = \cos(t), \quad \varphi_3(t) = \sin(t)$$

Determino $\quad \bar{\varphi} = \gamma_1 \varphi_1 + \gamma_2 \varphi_2 + \gamma_3 \varphi_3 \quad$ con

$\gamma_1, \gamma_2, \gamma_3$ tali che

$$\begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix} \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \gamma_3'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tan t \end{pmatrix} \quad \forall t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Calcolo:

$$\gamma_1'(t) = \frac{\begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \tan t & -\cos t & -\sin t \end{vmatrix}}{\begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix}} = \frac{\tan t \cdot 1}{1 \cdot 1} = \tan t$$

$$= \frac{\sin t}{\cos t} = -\frac{-\sin t}{\cos t}$$

Scelgo $\quad \gamma_1(t) = -\ln|\underbrace{\cos t}_{>0 \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}| = -\ln(\cos t)$

$$\gamma_2'(t) = \frac{\begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \tan t & -\sin t \end{vmatrix}}{1} = 1 \cdot (-\cos t \cdot \tan t)$$

$$= -\sin t$$

Scelgo $\quad \gamma_2(t) = \cos t$

$$\gamma_3'(t) = \frac{\begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \tan t \end{vmatrix}}{1} = -\sin t \cdot \tan t$$

$$= -\frac{\sin^2 t}{\cos t}$$

$$\int -\frac{\sin^2 t}{\cos t}\, dt = \int \frac{\cos^2 t - 1}{\cos t}\, dt = \int \cos t \, dt - \int \frac{1}{\cos t} dt$$

$$= \sin t - \int \frac{\cos t}{1-\sin^2 t}\, dt$$

$$= \sin t - \int \frac{1}{2}\left(\frac{1}{1-\sin t} + \frac{1}{1+\sin t}\right) \cos t \, dt$$

$$= \sin t - \frac{1}{2} \ln\left(\frac{1+\sin t}{1-\sin t}\right) + c$$

Scelgo $\gamma_3(t) = \sin t + \frac{1}{2} \ln\left(\frac{1-\sin t}{1+\sin t}\right)$

Quindi: una sol. particolare dell' eq. è

$$\bar{\varphi}(t) = \gamma_1(t)\varphi_1(t) + \gamma_2(t)\varphi_2(t) + \gamma_3(t)\varphi_3(t)$$

$$= -\ln(\cos t)\cdot 1 + \cos t \cdot \cos t +$$

$$+ \left(\sin t + \frac{1}{2}\ln\left(\frac{1-\sin t}{1+\sin t}\right)\right) \sin t$$

$$= -\ln(\cos t) + 1 + \frac{\sin t}{2} \ln\left(\frac{1-\sin t}{1+\sin t}\right)$$

Integrale generale:

$$c_1 \cdot 1 + c_2 \cos t + c_3 \sin t - \ln(\cos t) + 1 + \frac{\sin t}{2} \ln\left(\frac{1-\sin t}{1+\sin t}\right)$$

$$= c_1 + c_2 \cos t + c_3 \sin t - \ln(\cos t) + \frac{\sin t}{2} \ln\left(\frac{1-\sin t}{1+\sin t}\right) \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\square$

$c_1, c_2, c_3 \in \mathbb{R}$

Esempio

$$\begin{cases} y'' - 2y' + y = \ln(t) & t \in (0, +\infty) \\ y(1) = 0 \,, \quad y'(1) = 1 \end{cases}$$

$$P(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

SFS: $\varphi_1(t) = e^t \,, \quad \varphi_2(t) = te^t$

Metodo di somiglianza? No!

Metodo di Lagrange?

Cerco γ_1, γ_2 t.c.

$$\begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix} \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \ln(t) \end{pmatrix}$$

$$\gamma_1'(t) = \frac{\begin{vmatrix} 0 & te^t \\ \ln(t) & e^t + te^t \end{vmatrix}}{e^{2t}} = \frac{-te^t \ln(t)}{e^{2t}}$$

già calcolato $\rightarrow e^{2t}$

$$= -te^{-t}\ln(t) \quad ??$$

Determino il polinomio di Taylor di centro 1 e ordine 3 della soluzione del PdC.

Sia φ la soluzione.

$$T_{1,3}(t) = \underbrace{\varphi(1)}_{=0} + \underbrace{\varphi'(1)}_{=1}(t-1) + \frac{\varphi''(1)}{2}(t-1)^2 + \frac{\varphi'''(1)}{3!}(t-1)^3$$

φ è soluzione dell'eq. $\Rightarrow$

$$\forall t \in (0,+\infty): \quad \varphi''(t) - 2\varphi'(t) + \varphi(t) = \ln(t) \quad \Rightarrow$$

$$\forall t \in (0,+\infty): \quad \varphi''(t) = 2\varphi'(t) - \varphi(t) + \ln(t) \qquad (*)$$

$$\Rightarrow \varphi''(1) = 2\underbrace{\varphi'(1)}_{=1} - \underbrace{\varphi(1)}_{=0} + \underbrace{\ln(1)}_{=0} = 2$$

Da $(*)$: deduco che φ'' è derivabile

(quindi φ è derivabile TRE volte) e

$$\forall t \in (0,+\infty): \quad \varphi'''(t) = 2\varphi''(t) - \varphi'(t) + \frac{1}{t}$$

In particolare:

$$\varphi'''(1) = 2\varphi''(1) - \varphi'(1) + \frac{1}{1} = 4 - 1 + 1 = 4$$

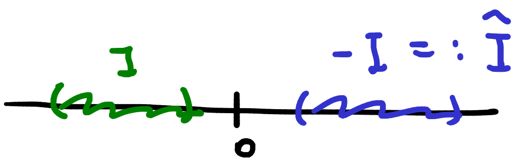
$$\Rightarrow T_{1,3}(t) = 0 + (t-1) + \frac{2}{2}(t-1)^2 + \frac{4}{3!}(t-1)^3$$

$$= (t-1) + t^2 - 2t + 1 + \frac{2}{3}t^3 - 2t^2 + 2t - \frac{2}{3}$$

$$= \frac{2}{3}t^3 - t^2 + t - \frac{2}{3}$$

Oss. (su eq. di Eulero)

Suppongo $I \subset (-\infty, 0)$



Definisco $\hat{I} := -I \quad (\subset (0,+\infty))$

$$\hat{b} : \hat{I} \to \mathbb{R} \quad \text{t.c.} \quad \hat{b}(t) = b(-t)$$

Suppongo che $\hat{\varphi} : \hat{I} \to \mathbb{R}$ sia soluzione dell'equazione con termine noto $\hat{b}$:

$$\forall t \in \hat{I}: \quad \sum_{k=0}^{n} a_k t^k \hat{\varphi}^{(k)}(t) = \hat{b}(t)$$

Questo equivale a:

$$\forall t \in I: \quad \sum_{k=0}^{n} a_k (-t)^k \hat{\varphi}^{(k)}(-t) = \hat{b}(-t) \qquad \odot$$

Definisco $\varphi : I \to \mathbb{R}$ t.c. $\varphi(t) = \hat{\varphi}(-t)$

Osservo che $\forall k$:

$$\varphi^{(k)}(t) = (-1)^k \hat{\varphi}^{(k)}(-t)$$

$$\odot \iff \forall t \in I: \quad \sum_{k=0}^{n} a_k t^k \underbrace{(-1)^k \hat{\varphi}^{(k)}(-t)}_{\varphi^{(k)}(t)} = \underbrace{\hat{b}(-t)}_{b(t)}$$

$\Rightarrow$ φ risolve in I l'eq. con termine noto b. $\square$

Esempi

- $t^2 y'' + 3t y' - 3y = 0 \quad (1) \qquad \underline{t > 0}$

$t = e^s, \quad s \in \mathbb{R} \iff s = \ln t, \quad t > 0$

Suppongo φ sia soluzione di (1).

Definisco $\psi(s) = \varphi(e^s), \quad s \in \mathbb{R}$

Equivale a: $\varphi(t) = \psi(\ln t)$, $t>0$.

Calcolo:

$$\varphi'(t) = \psi'(\ln t) \cdot \frac{1}{t}$$

$$\varphi''(t) = \psi''(\ln t) \cdot \frac{1}{t} \cdot \frac{1}{t} + \psi'(\ln t)\left(-\frac{1}{t^2}\right)$$

$$= \frac{\psi''(\ln t) - \psi'(\ln t)}{t^2}$$

φ è sol. di (1) $\iff$

$$\forall t>0: \quad t^2\varphi''(t) + 3t\,\varphi'(t) - 3\varphi(t) = 0 \iff$$

$$\forall t>0: \quad \cancel{t^2} \cdot \frac{\psi''(\ln t) - \psi'(\ln t)}{\cancel{t^2}} + 3\cancel{t}\,\frac{\psi'(\ln t)}{\cancel{t}} - 3\psi(\ln t) = 0$$

$$\iff \forall t>0: \quad \psi''(\ln t) - \psi'(\ln t) + 3\psi'(\ln t) - 3\psi(\ln t) = 0$$

$$\iff \forall t>0: \quad \psi''(\ln t) + 2\psi'(\ln t) - 3\psi(\ln t) = 0$$

$$\iff \forall s\in\mathbb{R}: \quad \psi''(s) + 2\psi'(s) - 3\psi(s) = 0$$

Cioè: ψ risolve in $\mathbb{R}$ l'equazione

$$\underbrace{z'' + 2z' - 3z = 0}_{\text{eq. a coefficienti costanti!}} \qquad (2)$$

Risolvo (2): $\quad P(\lambda) = \lambda^2 + 2\lambda - 3$

radici: $\lambda = 1$, $\quad \lambda = -3$

Integrale generale di (2):

$$c_1 e^{s} + c_2 e^{-3s}, \quad s \in \mathbb{R} \qquad (c_1, c_2 \in \mathbb{R})$$

Integrale generale di (1):

$$c_1 t + c_2 t^{-3}, \quad t > 0$$

(anche per $t < 0$)

($t = 0$ No!!)

- $\underbrace{t^2 y''}_{z''-z'} - 3\underbrace{t y'}_{z'} + 13 \underbrace{y}_{z} = 0 \qquad (1)$

Equazione ausiliaria: $\qquad z(s) = y(e^s)$

$$z'' - z' - 3z' + 13z = 0$$

$$z'' - 4z' + 13z = 0 \qquad (2)$$

$$P(\lambda) = \lambda^2 - 4\lambda + 13 \qquad \lambda = 2 \pm i3$$

Integr. gen. di (2):

$$c_1 e^{2s} \cos(3s) + c_2 e^{2s} \sin(3s), \quad s \in \mathbb{R}$$

$\Rightarrow$ Integr. generale di (1): $\qquad t = e^s, \quad s = \ln t$

$$c_1 t^2 \cos(3\ln(t)) + c_2 t^2 \sin(3\ln(t)), \quad t > 0$$

- $\underbrace{t^2 y''}_{z''-z'} - \underbrace{2ty'}_{z'} + \underbrace{2y}_{z} = t^3 \qquad (1)$

$t = e^s \,, \quad s = \ln t$

Eq. ausiliaria:

$$z'' - z' - 2z' + 2z = e^{3s}$$

$$z'' - 3z' + 2z = e^{3s} \qquad (2)$$

$$P(\lambda) = \lambda^2 - 3\lambda + 2 \qquad \lambda = 1,\ \lambda = 2$$

Cerco sol. partic. di (2) del tipo:

$$\overline{\psi}(s) = a e^{3s}$$

$$\dots \qquad a = \frac{1}{2}$$

$\Rightarrow$ Integr. gen. di (2):

$$c_1 e^s + c_2 e^{2s} + \frac{1}{2} e^{3s} \,, \qquad s \in \mathbb{R}$$

$\Rightarrow$ Integr. gen. di (1):

$$c_1 t + c_2 t^2 + \frac{1}{2} t^3 \,, \qquad t > 0$$

anche per $t < 0$

anche per $t = 0$