14 marzo 2024

$$P(\lambda) = \sum_{j=0}^{n} a_j \lambda^j = a_0 + a_1\lambda + \dots + a_n \lambda^n$$

$$P'(\lambda) = a_1 + 2a_2\lambda + \dots + n a_n \lambda^{n-1} = \sum_{j=1}^{n} a_j\, j\, \lambda^{j-1}$$

$$P''(\lambda) = \sum_{j=2}^{n} a_j\, j(j-1)\, \lambda^{j-2}$$

$$\vdots$$

$$P^{(k)}(\lambda) = k! \sum_{j=k}^{n} a_j \frac{j(j-1)\dots(j-k+1)}{k!} \lambda^{j-k}$$

? $\binom{j}{k}$

$f, g : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, derivabili

$$(fg)' = f'g + fg'$$

$$(fg)'' = f''g + 2f'g' + fg''$$

$$(fg)''' = f'''g + 3f''g' + 3f'g'' + fg'''$$

$$\vdots$$

Chiarimento ...

Es: $n = 2$

$$\sum_{j=0}^{2} \sum_{k=0}^{j} x_{jk} =$$

$$\sum_{k=0}^{0} x_{0k} + \sum_{k=0}^{1} x_{1k} + \sum_{k=0}^{2} x_{2k} =$$

$$x_{00} + x_{10} + x_{11} + x_{20} + x_{21} + x_{22}$$

$$\sum_{k=0}^{2} \sum_{j=k}^{2} x_{jk} =$$

$$\sum_{j=0}^{2} x_{j0} + \sum_{j=1}^{2} x_{j1} + \sum_{j=2}^{2} x_{j2} =$$

$$x_{00} + x_{10} + x_{20} + x_{11} + x_{21} + x_{22}$$

Esempi

- $y''' - y'' - y' + y = 0$

$$p(\lambda) = \lambda^3 - \lambda^2 - \lambda + 1 = \lambda^2(\lambda - 1) - (\lambda - 1)$$

$$= (\lambda - 1)(\lambda^2 - 1) = (\lambda - 1)^2 (\lambda + 1)$$

radici ; $\lambda = 1 \quad m = 2$

$\lambda = -1 \quad m = 1$

Soluzioni : $t \mapsto e^t , \quad t \mapsto te^t , \quad t \mapsto e^{-t}$

Verifico se sono lin. indip :

$$W(t) = \begin{pmatrix} e^t & te^t & e^{-t} \\ e^t & e^t + te^t & -e^{-t} \\ e^t & e^t + e^t + te^t & e^{-t} \end{pmatrix}$$

$$W(0) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\det(W(0)) = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 3 + 1 \neq 0 \checkmark$$

$$\Rightarrow \quad V_0 = \{ c_1 e^t + c_2 t e^t + c_3 e^{-t} , \; t \in \mathbb{R} \mid c_1, c_2, c_3 \in \mathbb{R} \}$$

- $y''' - 3y'' + 3y' - y = 0$

$P(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$

radici : $\lambda = 1 \qquad m = 3$

Soluzioni : $\quad e^t , \quad t e^t , \quad t^2 e^t$

[$\det(W(0)) \neq 0$?]

Chiarimento ...

$n = 3 \qquad e^{\lambda_1 t} \qquad t e^{\lambda_1 t} \qquad (m_1 = 2)$

$\qquad\qquad e^{\lambda_2 t} \qquad\qquad (m_2 = 1)$

$$\underbrace{c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}} + c_3 e^{\lambda_2 t} \equiv 0$$

$$\underbrace{(c_1 + c_2 t)}_{\substack{\text{polinomio} \\ \text{di grado } \leq 1}} e^{\lambda_1 t} + \underbrace{c_3}_{\substack{\text{pol.} \\ \text{di grado } 0}} e^{\lambda_2 t} \equiv 0$$

$\lambda = \alpha + i\beta$

$$t\, e^{\lambda t} = t e^{\alpha t} (\cos(\beta t) + i \sin(\beta t))$$

$$= \underbrace{t e^{\alpha t} \cos(\beta t)} + i \underbrace{t e^{\alpha t} \sin(\beta t)}$$

Esempi :

- $y''' + y = 0$

$$P(\lambda) = \lambda^3 + 1 = (\lambda+1)(\lambda^2 - \lambda + 1)$$

radici $\lambda = -1 \quad (m = 1)$

$$\lambda = \frac{1 \pm i\sqrt{3}}{2} \quad (m=1)$$

$$= \underbrace{\frac{1}{2}}_{\alpha} \pm i \underbrace{\frac{\sqrt{3}}{2}}_{\beta}$$

SFS :

$$e^{-t}, \quad e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), \quad e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$V_0 = \left\{ c_1 e^{-t} + c_2 e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right), \quad t \in \mathbb{R} \;\middle|\; c_1, c_2, c_3 \in \mathbb{R} \right\}$$

- $y^{(4)} + 4y'' + 4y = 0$

$$P(\lambda) = \lambda^4 + 4\lambda^2 + 4$$
$$= (\lambda^2 + 2)^2$$

Radici $i\sqrt{2}$, $-i\sqrt{2}$

con molteplicità 2

$$\left[\lambda^2 + 2 = 0 \iff \lambda = \pm i\sqrt{2} \right]$$

$$= (\lambda - i\sqrt{2})^2 (\lambda + i\sqrt{2})^2$$

$$\sqrt{2}\,i = \underbrace{0}_{\alpha} + i\underbrace{\sqrt{2}}_{\beta}$$

SFS : $\cos(\sqrt{2}t)$, $t\cos(\sqrt{2}t)$, $\sin(\sqrt{2}t)$, $t\sin(\sqrt{2}t)$

$$V_0 = \left\{ c_1\cos(\sqrt{2}t) + c_2 t\cos(\sqrt{2}t) + c_3\sin(\sqrt{2}t) + c_4 t\sin(\sqrt{2}t), \quad t \in \mathbb{R} \;\middle|\; c_1, c_2, c_3, c_4 \in \mathbb{R} \right\}$$

- $y^{(4)} + y''' - 3y'' - 5y' - 2y = 0$

$$P(\lambda) = \lambda^4 + \lambda^3 - 3\lambda^2 - 5\lambda - 2$$

$$P(-1) = 1 - 1 - 3 + 5 - 2 = 0 \quad \checkmark$$

$$\begin{array}{c|cccc|c}
 & 1 & 1 & -3 & -5 & -2 \\
-1 & & -1 & 0 & +3 & +2 \\
\hline
 & 1 & 0 & -3 & -2 & 0
\end{array}$$

$$\Rightarrow \quad P(\lambda) = (\lambda+1)(\underbrace{\lambda^3 - 3\lambda - 2}_{Q(\lambda)})$$

$$Q(1) = 1 - 3 - 2 \neq 0 \qquad Q(-1) = -1 + 3 - 2 = 0 \quad \checkmark$$

$$\begin{array}{c|ccc|c}
 & 1 & 0 & -3 & -2 \\
-1 & & -1 & +1 & +2 \\
\hline
 & 1 & -1 & -2 & 0
\end{array}$$

$$\Rightarrow \quad P(\lambda) = (\lambda+1)^2(\lambda^2 - \lambda - 2)$$

$$= (\lambda+1)^2(\lambda+1)(\lambda-2)$$

$$= (\lambda+1)^3(\lambda-2)$$

Radici: $\lambda = -1 \quad m = 3$

$\lambda = 2 \quad m = 1$

$$V_0 = \{ c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} + c_4 e^{2t}, \; t \in \mathbb{R} \mid c_1, c_2, c_3, c_4 \in \mathbb{R} \}$$

- $y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0$

$$P(\lambda) = \lambda^4 - 4\lambda^3 + \underbrace{8\lambda^2}_{4\lambda^2 + 4\lambda^2} - 8\lambda + 4$$

$$= (\lambda^2 - 2\lambda + 2)^2$$

$\lambda^2 - 2\lambda + 2 = 0$ $\qquad \lambda = 1 \pm i$

Radici : $\lambda = 1 + i$ $\qquad m = 2$

$\lambda = 1 - i$

$$V_0 = \{ c_1 e^t \cos(t) + c_2 e^t \sin(t) + c_3 t e^t \cos(t) + c_4 t e^t \sin(t),$$
$$t \in \mathbb{R} \mid c_1, c_2, c_3, c_4 \in \mathbb{R} \}$$

$$m y'' = -k y - \alpha y'$$

$$y'' + \underbrace{\frac{\alpha}{m}}_{>0 \; =: 2\gamma} y' + \underbrace{\frac{k}{m}}_{>0 \; =: \omega_0^2} y = 0$$

$y'' + 2\gamma y' + \omega_0^2 y = 0$ $\qquad \gamma \geq 0$

$\omega_0 > 0$

$P(\lambda) = \lambda^2 + 2\gamma\lambda + \omega_0^2$

1° caso : $\gamma = 0$ (no attrito)

$P(\lambda) = \lambda^2 + \omega_0^2$ $\qquad$ Radici : $\lambda = \pm i\omega_0$

Generica soluzione :

$$\varphi(t) = \underline{c_1 \cos(\omega_0 t)} + \underline{c_2 \sin(\omega_0 t)}$$

$$= A \sin(\omega_0 t + \theta)$$

$$= A\sin(\omega_0 t)\cos\theta + A\cos(\omega_0 t)\sin\theta$$

Scelgo A, θ in modo che

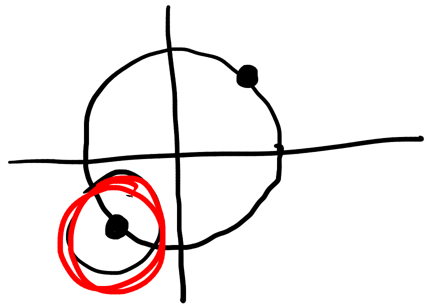
$$\begin{cases} A\sin\theta = c_1 \\ A\cos\theta = c_2 \end{cases} \qquad A^2 = c_1^2 + c_2^2 \implies A = \sqrt{c_1^2 + c_2^2}$$

$$\begin{cases} \sin\theta = \dfrac{c_1}{\sqrt{c_1^2 + c_2^2}} \\ \cos\theta = \dfrac{c_2}{\sqrt{c_1^2 + c_2^2}} \end{cases}$$

$$\tan\theta = \frac{c_1}{c_2} \qquad (c_2 \neq 0)$$

$$\theta = \arctan\left(\frac{c_1}{c_2}\right) \quad !!!$$

$\tan\theta = 1$?



Da completare ...