8 marzo 2024

Es : $\varphi_1(t) = t^2$, $\varphi_2(t) = t$

$$W(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \qquad \forall t \in (0, +\infty)$$

$$\Rightarrow \quad W(1) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\det(W(1)) = 1 - 2 = -1 \neq 0$$

$\Rightarrow$ φ_1 e φ_2 sono lin. indipendenti

L'integrale generale di:

$$y'' - \frac{2}{t} y' + \frac{2}{t^2} y = 0 \qquad t \in (0, +\infty)$$

è

$$V_0 = \{ c_1 \varphi_1 + c_2 \varphi_2 \mid c_1, c_2 \in \mathbb{R} \}$$
$$= \{ c_1 t^2 + c_2 t, \ t \in (0, +\infty) \mid c_1, c_2 \in \mathbb{R} \}$$

Verifico la Prop.

A : primitiva di a $\quad (\forall t \in I : A'(t) = a(t))$

$$\varphi_0(t) := e^{-A(t)}$$

- A derivabile $\overset{\text{exp deriv.}}{\Rightarrow}$ φ_0 derivabile ✓
- $\underline{\forall t \in I :}$

$$\underbrace{\varphi_0'(t) + a(t)\varphi_0(t) =}\ e^{-A(t)}(-A(t))' + a(t)\, e^{-A(t)}$$

$$= e^{-A(t)}(-a(t)) + a(t)\, e^{-A(t)}$$

$$= -\, e^{-A(t)} a(t) + a(t)\, e^{-A(t)} \underbrace{= 0}\ \checkmark$$

$\square$

- $y' + ty = 0$ $\quad a(t) = t \quad t \in \mathbb{R}$

Scelgo $A(t) = \dfrac{t^2}{2} \Rightarrow \varphi_0(t) = e^{-t^2/2}$, $t \in \mathbb{R}$

è soluzione dell' eq. diff.

L'integrale generale è

$$V_0 = \{ c\varphi_0 \mid c \in \mathbb{R} \} = \{ c e^{-t^2/2}, t \in \mathbb{R} \mid c \in \mathbb{R} \}$$

Risolvo il PdC con condizione iniziale

$$y(1) = 2$$

Pongo $\varphi_c(t) = c e^{-t^2/2}$ e determino c in modo che

$$\varphi_c(1) = 2$$

cioè:

$$c e^{-1/2} = 2 \iff c = 2 e^{1/2} = 2\sqrt{e}$$

Conclusione: la sol. del PdC è

$$\varphi(t) = 2\sqrt{e}\, e^{-t^2/2}, \quad t \in \mathbb{R}$$

$\square$

• $y' - \sin(t)\, y = 0$ $\quad a(t) = -\sin(t)\,,\ t \in \mathbb{R}$

Scelgo $A(t) = \cos(t)$

$\Rightarrow\ \varphi_0(t) = e^{-\cos(t)}\,,\ t \in \mathbb{R}$ sol. dell'eq. diff.

$$\Rightarrow\ V_0 = \{ \underbrace{c\,e^{-\cos(t)}}_{\varphi_c(t)}\,,\ t \in \mathbb{R} \mid c \in \mathbb{R} \}$$

Risolvo il PdC con $y(\frac{\pi}{2}) = -1$:

$$\varphi_c\left(\tfrac{\pi}{2}\right) = -1 \iff c\,\underbrace{e^{-\overbrace{\cos(\frac{\pi}{2})}^{=0}}}_{=1} = -1 \iff c = -1$$

Soluzione : $\varphi(t) = -e^{-\cos(t)}\,,\ t \in \mathbb{R}$ $\square$

• $y' - e^{t^2} y = 0$ $\quad a(t) = -e^{t^2}\,,\ t \in \mathbb{R}$

Scelgo $A(t) =$???

non posso determinarla esplicitamente in termini di funzioni elementari

$$A(t) = \int_0^t a(s)\, ds = \int_0^t (-e^{s^2})\, ds = -\int_0^t e^{s^2}\, ds$$

$\Rightarrow\ \varphi_0(t) = e^{\int_0^t e^{s^2} ds}\,,\ t \in \mathbb{R}$ è sol. dell'eq. diff.

$$\Rightarrow\ V_0 = \{ \underbrace{c\,e^{\int_0^t e^{s^2} ds}}_{\varphi_c(t)}\,,\ t \in \mathbb{R} \mid c \in \mathbb{R} \}$$

Risolvo il PdC con $y(0) = -2$

$$\varphi_c(0) = -2 \iff c\,\underbrace{e^{\overbrace{\int_0^0 e^{s^2} ds}^{=0}}}_{=1} = -2 \iff c = -2$$

Soluzione : $\varphi(t) = -2\,e^{\int_0^t e^{s^2} ds}\,,\ t \in \mathbb{R}$ $\square$

$$\tilde{A}(t) = -\int_7^t e^{s^2}\,ds$$

"Cosa succede se scelgo in A un punto iniziale diverso da 0"

$$\tilde{\varphi}_0(t) = e^{\int_7^t e^{s^2}\,ds}$$

$$\tilde{\varphi}_c(t) = c\, e^{\int_7^t e^{s^2}\,ds}$$

$$\tilde{\varphi}_c(0) = -2 \iff c\, e^{\int_7^0 e^{s^2}\,ds} = -2$$

$$\iff c = -2\, e^{-\int_7^0 e^{s^2}\,ds} = -2\, e^{\int_0^7 e^{s^2}\,ds}$$

Sol: $\tilde{\varphi}(t) = -2\, e^{\int_0^7 e^{s^2}\,ds} \cdot e^{\int_7^t e^{s^2}\,ds}$

$$= -2\, e^{\int_0^7 e^{s^2}\,ds + \int_7^t e^{s^2}\,ds}$$

$$= -2\, e^{\int_0^t e^{s^2}\,ds} = \underline{\underline{\varphi(t)}}$$

- $y' + \tan(t)\, y = 0$

$a(t) = \tan(t)$

$t \in \underbrace{\mathbb{R} \setminus \{ \frac{\pi}{2} + k\pi \mid k \in \mathbb{Z} \}}_{\text{non è un intervallo}}$

Tenuto conto della condizione iniziale

$y(\pi) = 3$,

scelgo $I = \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$

Considero l'equazione in

$\left(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right)$

Cerco una primitiva di: $a(t) = \tan(t)$ in $\left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$

Scelgo: $A(t) = -\ln|\cos(t)|$

$$\Rightarrow \quad \varphi_0(t) = e^{-A(t)} = e^{\ln|\cos(t)|} = |\cos(t)|$$

$$\overrightarrow{t \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)} = -\cos(t)$$

$$\Rightarrow \quad V_0 = \left\{ c(-\cos(t)),\ t \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right) \ \middle|\ c \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{c \cos(t)}_{\varphi_c(t)},\ t \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right) \ \middle|\ c \in \mathbb{R} \right\}$$

Risolvo il PdC :

$$\varphi_c(\pi) = 3 \iff c\cos(\pi) = 3 \iff -c = 3$$
$$\iff c = -3$$

Soluzione : $\varphi(t) = -3\cos(t), \quad t \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$ $\square$

- $y' + ay = 0 \qquad a \in \mathbb{R} \qquad t \in \mathbb{R}$

Scelgo $A(t) = at$

$$\Rightarrow \quad \varphi_0(t) = e^{-at}, \quad t \in \mathbb{R}$$

$$\Rightarrow \quad V_0 = \left\{ c e^{-at},\ t \in \mathbb{R} \ \middle|\ c \in \mathbb{R} \right\}$$

$$\left[\ y' + 3y = 0 \qquad y' = -3y\ \right]$$

Equaz. a coeff. costanti :

$$y^{(n)} + \underbrace{a_{n-1}}_{\in \mathbb{R}} y^{(n-1)} + \dots + \underbrace{a_1}_{\in \mathbb{R}} y' + \underbrace{a_0}_{\in \mathbb{R}} y = 0 \qquad (*)$$

$\varphi(t) := e^{\lambda t}, \quad t \in \mathbb{R}$

Calcolo

$$L(\varphi)(t) = \underbrace{\varphi^{(n)}(t)}_{= \lambda^n e^{\lambda t}} + a_{n-1} \underbrace{\varphi^{(n-1)}(t)}_{= \lambda^{n-1} e^{\lambda t}} + \ldots + a_1 \underbrace{\varphi'(t)}_{= \lambda e^{\lambda t}} + a_0 \varphi(t)$$

$\varphi''(t) = \lambda^2 e^{\lambda t}$

$$= \lambda^n e^{\lambda t} + a_{n-1} \lambda^{n-1} e^{\lambda t} + \ldots + a_1 \lambda e^{\lambda t} + a_0 e^{\lambda t}$$

$$= e^{\lambda t} \left(\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 \right) \quad \forall t \in \mathbb{R}$$

Quindi:

φ risolve $(*)$ $\iff$

$\forall t \in \mathbb{R} : \quad L(\varphi)(t) = 0 \iff$

$$\forall t \in \mathbb{R} : \quad \underbrace{e^{\lambda t}}_{\neq 0} \left(\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 \right) = 0 \iff$$

$$\cancel{\forall t \in \mathbb{R}} : \quad \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$$

Es:

$y'' - y = 0 \quad \odot$

Eq. caratteristica: $\lambda^2 - 1 = 0$

Soluzioni: $\lambda = 1, \quad \lambda = -1$

$\Rightarrow \quad \varphi(t) = e^{t}, \quad \psi(t) = e^{-t}$ sol. di $\odot$

Verifico se φ e ψ sono lin. indip:

$$W(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$$

$$\det(W(t)) = e^t(-e^{-t}) - e^{-t}(e^t) = -1-1 = -2 \neq 0$$

$\Rightarrow$ φ, ψ lin. indip.

$$\Rightarrow V_0 = \{ c_1 e^t + c_2 e^{-t}, \ t \in \mathbb{R} \mid c_1, c_2 \in \mathbb{R} \}$$

- $y'' - 3y' + 2y = 0$ $\odot$

$$\lambda^2 - 3\lambda + 2 = 0 \qquad \lambda = 1, \quad \lambda = 2$$

$$\left(\lambda_{1/2} = \frac{3 \mp \sqrt{9-8}}{2} \right)$$

$\Rightarrow$ $\varphi_1(t) = e^t$, $\varphi_2(t) = e^{2t}$ sol. di $\odot$

$$W(t) = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}$$

$$W(0) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}; \quad \det(W(0)) = 2-1 \neq 0 \ \checkmark$$

$$V_0 = \{ c_1 e^t + c_2 e^{2t}, \ t \in \mathbb{R} \mid c_1, c_2 \in \mathbb{R} \}$$

- $y'' + 4y' + 4y = 0$ $\odot$

$$P(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

Una sola radice con molteplicità 2 : $\lambda = -2$

$\Rightarrow \ \varphi(t) = e^{-2t}, \ t \in \mathbb{R}$ è sol. di: $\odot$

$V_0 = ???$

una sola soluzione non basta a generare V_0 che ha dimensione $\underline{\underline{2}}$

- $y'' + y = 0$

$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$!!!

Es: $\lambda = \alpha + i\beta, \quad \alpha \in \mathbb{R}, \ \beta \in \mathbb{R}^*$

$$z(t) = e^{\lambda t} := e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

$$= \underbrace{e^{\alpha t}\cos(\beta t)}_{=: u(t)} + i \underbrace{e^{\alpha t}\sin(\beta t)}_{=: v(t)}$$

u, v derivabili (perché lo sono exp, sin, cos)

$\overset{def}{\Rightarrow}$ z è derivabile.

Calcolo

$$z'(t) \overset{def}{=} u'(t) + i v'(t)$$

$$= \alpha e^{\alpha t}\cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) + i\left(\alpha e^{\alpha t}\sin(\beta t) + \beta e^{\alpha t}\cos(\beta t)\right)$$

$$= \alpha\left[e^{\alpha t}\cos(\beta t) + i e^{\alpha t}\sin(\beta t)\right] + \beta\left[i e^{\alpha t}\cos(\beta t) \underbrace{-}_{= i^2} e^{\alpha t}\sin(\beta t)\right]$$

$$= \alpha\left[e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\right] + i\beta\left[e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\right]$$

$$= \alpha e^{\lambda t} + i\beta e^{\lambda t}$$

$$= (\alpha + i\beta)\, e^{\lambda t} = \lambda e^{\lambda t} \qquad \square$$

Es: $z(t) = e^{it} \qquad t \in \mathbb{R}$ $\qquad \lambda = i \qquad (\alpha = 0, \beta = 1)$

$z'(t) = i e^{it}$

$z''(t) = i^2 e^{it} = -e^{it}$

$z''(t) + z(t) = -e^{it} + e^{it} = 0 \qquad \forall t \in \mathbb{R} \quad \checkmark$

z risolve $\quad y'' + y = 0$

$$\Rightarrow \quad z(t) = e^{it} = \underbrace{\cos(t)}_{u(t)} + i\, \underbrace{\sin(t)}_{v(t)}$$

$t \mapsto \cos(t)\,, \qquad t \mapsto \sin(t)$

sono soluzioni di $\quad y'' + y = 0$