

Storia e Fondamenti della Matematica
a.a. 2019/2020

Traccia d'esame – Aprile 2021 - 1

Una nuova visione della matematica pura, la cui idea fondante scaturisce dalle esigenze linguistiche e operative delle scienze empiriche: nell'analizzare il testo qui accluso, estratto dal volume *Lectures on Quaternions* di William Rowan Hamilton (1853), si trattino i seguenti aspetti, anche con riferimento ad altre opere di questo o di altri autori, secondo una prospettiva storica:

- la distinzione fra algebra, aritmetica e geometria;
- il ruolo di analisi e sintesi;
- l'astronomia come fonte di ispirazione per lo sviluppo di teorie matematiche;
- la misura del tempo e il concetto di moto nell'evoluzione del pensiero scientifico.

PREFACE.

[1.] THE volume now offered to the public is designed as an assistance to those persons who may be disposed to study and to employ a certain new mathematical method, which has, for some years past, occupied much of my own attention, and for which I have ventured to propose the name of the Method or Calculus of Quaternions. Although a copious analytical index, under the form of a Table of Contents, will be found to have been prefixed to the work, yet it seems proper to offer here some general and preliminary* remarks: especially as regards that conception from which the whole has been gradually evolved, and the motives for giving to the resulting method an appellation not previously in use.

[2.] The difficulties which so many have felt in the doctrine of Negative and Imaginary Quantities in Algebra forced themselves long ago on my attention; and although I early formed some acquaintance with various views or suggestions that had been proposed by eminent writers, for the purpose of removing

*Some readers may find it convenient to pass over for the present these prefatory remarks, and to proceed at once to the Volume, of which a large part has been drawn up so as to suppose less of previous and technical preparation than some of the paragraphs of this Preface. Indeed, great pains have been taken to render the early Lectures as elementary as the subject would allow; and it is hoped that they will be found perfectly and even easily intelligible by persons of moderate scientific attainments. It is true that some of the subsequent portions of the Course (especially parts of the concluding Lecture) may possibly appear difficult, from the novel nature of the calculations employed: but perhaps on that very account those later portions may repay the attention of more advanced mathematical students.

or eluding those difficulties (such as the theory of direct and inverse quantities, and of indirectly correlative figures, the method of constructing imaginaries by lines drawn from one point with various directions in one plane, and the view which refers all to the mere play of algebraical operations, and to the properties of symbolical language), yet the whole subject still appeared to me to deserve additional inquiry, and to be susceptible of a more complete elucidation. And while agreeing with those who had contended that negatives and imaginaries were not properly *quantities* at all, I still felt dissatisfied with any view which should not give to them, from the outset, a clear interpretation and *meaning*; and wished that this should be done, for the square roots of negatives, without introducing considerations *so expressly geometrical*, as those which involve the conception of an *angle*.

[3.] It early appeared to me that these ends might be attained by our consenting to regard ALGEBRA as being no mere Art, nor Language, nor *primarily* a Science of Quantity; but rather as the Science of Order in Progression. It was, however, a part of this conception, that the *progression* here spoken of was understood to be *continuous* and *unidimensional*: extending indefinitely *forward* and *backward*, but not in any *lateral* direction. And although the successive *states* of such a progression might (no doubt) be represented by *points upon a line*, yet I thought that their simple *successiveness* was better conceived by comparing them with *moments of time*, divested, however, of all reference to *cause* and *effect*; so that the “time” here considered might be said to be abstract, ideal, or *pure*, like that “space” which is the object of geometry. In this manner I was led, many years ago, to regard Algebra as the SCIENCE OF PURE TIME: and an Essay,* containing my views respecting it as such, was published† in 1835. If I now reproduce a few of the opinions put

* Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time. (Read November 4th, 1833, and June 1st, 1835).—Transactions of the Royal Irish Academy, Vol. XVII., Part II. (Dublin, 1835), pages 293 to 422.

† I was encouraged to entertain and publish this view, by remembering some passages in Kant's Criticism of the Pure Reason, which appeared to justify the expectation that it should be *possible* to construct, *à priori*, a Science of Time,

forward in that early Essay, it will be simply because they may assist the reader to place himself in that *point of view*, as regards the first elements of *algebra*, from which a passage was gradually made by me to that comparatively *geometrical* conception which it is the aim of this volume to unfold. And with respect to anything unusual in the *interpretations* thus proposed, for some simple and elementary notations, it is my wish to be understood as not at all insisting on them as *necessary*,* but merely proposing them as consistent among themselves, and preparatory to the study of the quaternions, in at least one aspect of the latter.

[4.] In the view thus recently referred to, if the letters *A* and *B* were employed as *dates*, to denote any two *moments* of time, which might or might not be distinct, the case of the coincidence or *identity* of these two moments, or of *equivalence* of these two dates, was denoted by the equation,

$$B = A;$$

which symbolic assertion was thus interpreted as not involving any *original* reference to *quantity*, nor as expressing the result

as well as a Science of Space. For example, in his Transcendental Æsthetic, Kant observes:—"Zeit und Raum sind demnach zwey Erkenntnisquellen, aus denen *à priori* verschiedene synthetische Erkenntnisse geschöpft werden können, wie vornehmlich die reine Mathematik in Ansehung der Erkenntnisse vom Raume und dessen Verhältnissen ein glänzendes Beyspiel gibt. Sie sind nämlich beide zusammengenommen reine Formen aller sinnlichen Anschauung, und machen dadurch synthetische Sätze *a priori* möglich." Which may be rudely rendered thus:—"Time and Space are therefore two knowledge-sources, from which different synthetic knowledges can be *à priori* derived, as eminently in reference to the knowledge of space and of its relations a brilliant example is given by the pure mathematics. For they are, both together [space and time], pure forms of all sensuous intuition, and make thereby synthetic positions *à priori* possible." (Critik der reinen Vernunft, p. 41. Seventh Edition. Leipzig: 1828).

* For example, the usual identity $(B - A) + A = B$, which in the older Essay was interpreted with reference to *time*, as in paragraph [8] of this Preface, the letters *A* and *B* denoting *moments*, is in the present work (Lecture I., article 25) interpreted, on an analogous plan indeed, but with a reference to *space*, the letters denoting *points*. Still it will be perceived that there exists a close connexion between the two views; a *step*, in each, being conceived to be applied to a *state* of a progression, so as to generate (or conduct to) another state. And generally I think that it may be found useful to compare the interpretations of which a sketch is given in the present Preface, with those proposed in the body of the work.

of any comparison between two *durations* as *measured*. It corresponded to the conception of simultaneity or *synchronism*; or, in simpler words, it represented the thought of the *present* in time. Of all possible answers to the general question, "*When*," the *simplest* is the answer, "*Now*:" and it was the *attitude of mind*, assumed in the making of this answer, which (in the system here described) might be said to be originally symbolized by the *equation* above written. And, in like manner, the two formulæ of *non-equivalence*,

$$B > A, B < A,$$

were interpreted, without any *primary* reference to quantity, as denoting the two contrasted relations of *subsequence* and of *precedence*, which answer to the thoughts of the *future* and the *past* in time; or as expressing, simply, the one that the moment B is conceived to be *later* than A, and the other that B is *earlier* than A: without *yet* introducing even the *conception* of a *measure*, to determine *how much later*, or how much earlier, one moment is than the other.

[5.] Such having been proposed as the *first* meanings to be assigned to the three elementary marks $= > <$, it was next suggested that the *first* use of the mark $-$, in constructing a *science of pure time*, might be conceived to be the forming of a complex symbol $B - A$, to denote the *difference between two moments*, or the *ordinal relation* of the moment B to the moment A, whether that relation were one of identity or of diversity; and if the latter, then whether it were one of subsequence or of precedence, and in whatever degree. And *here*, no doubt, in attending to the *degree* of such diversity between two moments, the conception of *duration*, as *quantity* in time, was introduced: the *full* meaning of the symbol $B - A$, in any particular application, being (on this plan) not known, until we know *how long after*, or how long before, if at all, B is than A. But it is evident that the notion of a certain *quality* (or *kind*) of this diversity, or interval, enters into this conception of a *difference* between moments, at least as fully and as soon as the notion of *quantity*, amount, or duration. The contrast between the Future and the Past appears to be even earlier and more fundamental, in human thought, than that between the Great and the Little.

[6.] After *comparing moments*, it was easy to proceed to *compare relations*; and in this view, by an *extension* of the recent signification [4] of the sign =, it was used to denote *analogy* in time; or, more precisely, to express the *equivalence of two marks of one common ordinal relation*, between *two pairs* of moments. Thus the formula,

$$D - C = B - A,$$

came to be interpreted as denoting an *equality between two intervals in time*; or to express that the moment D is *related* to the moment C, *exactly as* B is to A, with respect to identity or diversity: the *quantity and quality* of such diversity (when it exists) being here *both* taken into account. A formula of this sort was shewn to admit of *inversion* and *alternation* ($C - D = A - B$, $D - B = C - A$); and generally there could be performed a number of *transformations* and *combinations* of equations such as these, which all admitted of being *interpreted* and *justified* by this mode of viewing the subject, but which *agreed* in all respects with the received *rules* of algebra. On the same plan, the two contrasted formulæ of inequalities of differences,

$$D - C > B - A, \quad D - C < B - A,$$

were interpreted as signifying, the one that D was *later, relatively* to C, than B to A; and the other that D was *relatively earlier*.

[7.] Proceeding to the mark +, I used this sign *primarily* as a mark of combination between a symbol, such as the smaller Roman letter a, of a *step in time*, and the symbol, such as A, of the moment *from* which this *step* was conceived to be made, in order to form a complex symbol, $a + A$, *recording this conception of transition*, and denoting the moment (suppose B) *to* which the step was supposed to conduct. The step or transition here spoken of was regarded as a *mental act*, which might as easily be supposed to conduct *backwards* as *forwards* in the progression of time; or even to be a *null step*, denoted by 0, and producing *no effect* ($0 + A = A$). Thus, with these meanings of the signs, the notation

$$B = a + A,$$

denoted the conception that the moment B might be *attained*, or

mentally *generated*, by making (in thought) the step a from the moment A . And it appeared to me that without ceasing to regard the symbol $B - A$ as denoting, in one view [5], an *ordinal relation* between two moments, we might *also* use it in the *connected sense* of denoting this *step from one to another*: which would allow us (as in ordinary algebra) to write, with the recent suppositions,

$$B - A = a ;$$

the two members of this new equation being here symbols for one common step.

ON QUATERNIONS.

LECTURE I.

GENTLEMEN,

IN the preceding Lectures of the present Term, we have taken a rapid view of the chief facts and laws of Astronomy, its leading principles and methods and results. After some general and preliminary remarks on the connexion between metaphysical and physical science, we have seen how the observation of the elementary phenomena of the Heavens may be assisted, and rendered more precise, by means of astronomical instruments, accompanied with astronomical reductions. An outline of Uranography has been given; the laws of Kepler for the Solar System have been stated and illustrated; with the inductive evidence from facts by which their truth may be established. It has been shewn that these laws extend, not only to the Planets known in Kepler's time, namely, Mercury, Venus, Mars, Jupiter, and Saturn, with which our Earth must be enumerated, but also to the various other planets since detected: to Uranus, to Ceres, Pallas, Juno, and Vesta; and to those others of more recent date, in the order of human knowledge, of which no fewer than six have been found within the last two years and a half; to Astræa, Neptune, Hebe, Iris, Flora, and Metis: among which Neptune is remarkable, as having had its existence foreshewn by mathematical calculation, and Metis is interesting to us Irishmen, as having been discovered at an Irish observatory. It has also been shewn you that these celebrated laws of Kepler are themselves mathematically included in one still greater Law, with which the name of Newton is associated: and that thus, as New-

ton himself demonstrated, in his immortal work, the *Principia*, the rules of the elliptic motion of the planets are consequences of the principle of universal Gravitation, proportional directly to the mass, and inversely to the square of the distance. With the help of this great principle, or law, of Newton's, combined with proper observations and experiments,—especially, with the Cavendish experiment, as lately repeated by Baily,—not only have the *shape* and *size* of the earth which we inhabit, but even (as you have seen explained and illustrated) its very *weight* has been determined; the number of millions of millions of millions of tons of matter, which this vast globe contains, has been (approximately) assigned. And not only have the motions of that Earth of our's around and with its own axis, and round the sun, been established, but that great central body of our system, the Sun, through the persevering application of those faculties which God has given to man, has itself (as you have likewise seen) been measured and weighed, with the line and balance of science.

2. Such having been our joint contemplations in this place, before the adjournment of these discourses on account of the Examinations for Fellowships, you may remember that it was announced that at our re-assembling we should proceed to the consideration of a certain new mathematical Method, or Calculus, which has for some years past occupied a large share of my own attention, but which I have hitherto abstained from introducing, except by allusion, to the notice of those who have honoured here my lectures with their attendance. I refer, as you are aware, to what I have called the CALCULUS of QUATERNIONS, and have applied to the solution of many geometrical and physical problems.

4. The object which I shall propose to myself, in the Lecture of this day, is the statement of the significations, at least the *primary* significations, which I attach, in the Calculus of Quaternions, to the four following familiar marks of combination of symbols,

$$+ \quad - \quad \times \quad \div$$

which marks, or signs, are universally known to correspond, in arithmetic and in ordinary algebra, to the four *operations* known by the names of Addition, Subtraction, Multiplication, and Division. The *new* significations of these four signs have a sufficient *analogy* to the *old* ones, to make me think it convenient to *retain the signs* themselves; and yet a sufficient *distinction* exists, to render a *preliminary comment* not superfluous: or rather it is *indispensable* that as clear a definition, or at least *exposition*, of the precise force of each of these old marks, used in new senses, should be given, as it is in my power to give. Perhaps, indeed, I may not find it possible, to-day, to speak with what may seem the requisite degree of *fulness* of such exposition, of more than the *two first* of these four signs; although I hope to touch upon the two last of them also.

5. First, then, I wish to be allowed to say, in *general* terms (though conscious that they will need to be afterwards particularized), that I regard the two connected but contrasted marks or signs,

$$+ \text{ and } -,$$

as being respectively and *primarily characteristics of the SYNTHESIS and ANALYSIS of a STATE of a Progression*, according as this state is considered as being *derived from*, or *compared with*, some *other state* of that progression. And, with the same kind of generality of expression, I may observe here that I regard in like manner the *other pair* of connected and contrasted marks already mentioned, namely,

\times and \div ,

(when taken in what I look upon as their respectively *primary* significations), as being signs or characteristics of the corresponding SYNTHESIS and ANALYSIS of a STEP, in any such progression of states, according as that *step* is considered as *derived from*, or *compared with*, some *other step* in the same progression. But I am aware that this very general and preliminary statement cannot fail to appear vague, and that it is likely to seem also obscure, until it is rendered precise and clear by examples and illustrations, which the plan of these Lectures requires that I should select from Geometry, while it allows me to clothe them in an Astronomical garb. And I shall begin by endeavouring thus to illustrate and exemplify the view here taken of the sign $-$, which we may continue to *read*, as usual, MINUS, although the operation, of which it is now conceived to direct the performance, is not to be confounded with arithmetical, nor even, in *all* respects, with common algebraical subtraction.

6. I have said that I regard, *primarily*, this sign,

$-$, or Minus,

as the mark or characteristic of an *analysis of one state* of a progression, when considered as *compared with another* state of that progression. To illustrate this very general view, which has been here propounded, at first, under a metaphysical rather than a mathematical form, by proceeding to apply it under the limitations which the science of *geometry* suggests, let SPACE be now regarded as the *field* of the progression which is to be studied, and POINTS as the *states* of that progression. You will then see that in conformity with the general view already enunciated, and as its geometrical particularization, I am led to regard the word "Minus," or the mark $-$, in geometry, as the sign or

characteristic of the analysis of one geometrical position (in space), as compared with another (such) position. The *comparison of one mathematical point with another*, with a view to the determination of what may be called their *ordinal relation*, or their *relative position* in space, is in fact the investigation of the GEOMETRICAL DIFFERENCE of the two points compared, in that *sole* respect, namely, *position*, in which two mathematical points *can differ* from each other. And even for this reason alone, although I think that other reasons will offer themselves to your own minds, when you shall be more familiar with this whole aspect of the matter, you might already grant it to be *not unnatural* to regard, as it has been stated that I *do* regard, this study or investigation of the relative position of two points in space, as being that *primary geometrical operation* which is *analogous to algebraic subtraction*, and which I propose accordingly to denote by the usual mark $(-)$ of the well-known operation last mentioned. Without pretending, however, that I have yet exhibited sufficiently *conclusive* grounds for believing in the existence of such an *analogy*, I shall now proceed to illustrate, by *examples*, the modes of symbolical *expression* to which this belief, or view, conducts.

7. To illustrate first, by an astronomical example, the conception already mentioned, of the analysis of one geometrical position considered with reference to another, I shall here write down, as symbols for the two positions in space which are to be compared among themselves, the astronomical signs,

☉ and ♂ ;

which represent or denote respectively the sun and earth, and are *here* supposed to signify, *not* the masses, nor the longitudes, of those two bodies, nor any other *quantities* or magnitudes connected with them, *but simply their SITUATIONS*, or the positions of their centres, regarded as mathematical POINTS in space. To make more manifest to the eye that these astronomical signs are here employed to denote points or positions alone, I shall write under each a *dot*, and under the dot a Roman capital letter, namely, A for the earth, and B for the sun, as follows :

☉	♂	
·	·	
B	A	(Fig. 1.)

and shall suppose that the particular operation of what we have already called analysis, using that word in a very general and rather in a metaphysical than in a mathematical sense, which is now to be performed, consists in the proposed *investigation of the position of the sun, B, with respect to the earth, A*; the latter being regarded as comparatively simple and known; but the former as complex, or at least unknown and undetermined; and a relation being sought, which shall connect the one with the other.