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La matematica al centro di una visione filosofico-religiosa ed enciclopedica del mondo. Nell'analizzare il presente brano, tratto dalla traduzione inglese del trattato medievale latino *De Institutione Arithmetica* del filosofo cristiano Severino Boezio (480-524/526), si evidenzino i seguenti elementi, anche alla luce di altre opere, di questo o di altri autori:

- la classificazione delle scienze e i rapporti gerarchici tra gli oggetti della conoscenza;
- le influenze pitagoriche ed euclidee nel concetto di numero;
- le nozioni di finito/infinito e limitato/illimitato in aritmetica e geometria;
- l'unità e la distinzione tra pari e dispari.

Boethian Number Theory

A Translation of the *De Institutione Arithmetica*

(with Introduction and Notes)

by

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Here begin the Chapter Titles of the First Book

Here end the chapter titles of the first book.

BOETHIUS

De Institutione Arithmetica

Here begins the *First Book*.

1. Proemium: the division of mathematics.

Among all the men of ancient authority who, following the lead of Pythagoras, have flourished in the purer reasoning of the mind, it is clearly obvious that hardly anyone has been able to reach the highest perfection of the disciplines of philosophy unless the nobility of such wisdom was investigated by him in a certain four-part study, the *quadrivium*, which will hardly be hidden from those properly respectful of expertness.[3] For this is the wisdom of things which are, and the perception of truth gives to these things their unchanging character.

We say those things *are* which neither grow by stretching nor diminish by crushing, nor are changed by variations, but are always in their proper force and keep themselves secure by support of their own nature. Such things are: qualities, quantities, configurations, largeness, smallness, equalities, relations, acts, dispositions, places, times, and whatever is in any way found joined to bodies.[4] Now those things which by their nature are incorporeal, existing by reason of an immutable substance, when affected by the participation of a body and by contact with some variable thing, pass into a condition of inconstant changeableness. Such things (since as it was said, immutable substance and forces were delegated by nature) are truly and properly said to be. Wisdom gives name to a science

3. The first chapter of this book is taken from the *Arithmetica* of Nicomachus, Book 1, chap. 1-5. That work appears in English as *Introduction to Arithmetic* trans. M.L. D'ogge, intro. F. E. Robbins and L.C. Karpinski (New York, Macmillan, 1926). Nicomachus however, does not mention a four-fold way; Boethius here coins the term so important in the medieval Liberal Arts curriculum. The earliest manuscripts consistently use the version *quadruvium*, and *quadrivium* is a later spelling.
4. Boethius speaks of the ideas of these categories, distingishing them from particular occurrences. For him, as for the earlier Pythagoreans, they are distinguished from material, bodily things, subject to change and therefore not real.

in terms of these things, that is, things which properly exist, whatever their essences may be.[5]

There are two kinds of essence. One is continous, joined together in its parts and not distributed in separate parts, as a tree, a stone, and all the bodies of this world which are properly called magnitudes.[6] The other essence is of itself disjoined and determinded by its parts as though reduced to a single collective union, such as a flock, a populace, a chorus, a heap of things, things whose parts are terminated by their own extremities and are discrete from the extremity of some other. The proper name for these is a multitude. Again, some types of multitude exist by themselves, as a three, a four, a tetragon, or whatever number which, as it is, lacks nothing. Another kind does not exist of itself but refers to some other thing, as a duplex, a dimidium, a sesquialtar, a sesquitertial,[7] or whatever it may be which, unless it is related to anything, is not in itself able to exist. Of magnitudes, some are stable, lacking in motion, while others are always turned in mobile change and at no time are at rest.[8] Now of these types, arithmetic considers that multitude which exists of itself as an integral whole; the measures of musical modulation understand that multitude which exists in relation to some other; geometry offers the notion of stable magnitude; the skill of astronomical discipline explains the science of moveable magnitude. If a searcher is lacking knowledge of these four sciences, he is not

5. Philosophy etymologically means the love and study of such wisdom. The use of the word »philosopher« (one who pursues wisdom) was first ascribed to Pythagoras. See Diogenes Laertius, *De Vita et Moribus Philosophorum*, Book 1, chap.12.

6. On the distinction between multiude and magnitude, see Proclus, *In Primum Euclidis · Elementorum Comentarii*, ed. G. Friedlein (Leipzig, Teubner, 1873), p. 36 and Jcob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge, Mass., M.I.T. Press, 1968), pp. 10-11.

7. Duplex, a relation of 1:2; sesquialtar, 2:3; sesquitertial, 3:4.

8. The division of mathematical sciences into four branches comes to Boethius from Nicomachus and may be outlined in this fashion:

I. Science of number (a) As such, absolutely *καθ᾽ ἑαυτό* *Arithmetic*
(b) Relatively
πρὸς ἄλλο *Music*

II. Science of quantity (a) At rest *ἠρεμοῦν* *Geometry* (b) In motion *σφαιρική* *Astronomy*

A similar sivision is found in Proclus. Theon, however, includes music under arithmetic, maintaining they are identical sciences. There are instances in Boethius' treatise where this distinction does not hold, and problems arise between the limits for the first and second disciplines. Euclid and Domnius (5th century) avoid the distinction. For discussions, see Theon of Smyrna, *Exposition des Connaissances Mathématiques Utiles pour la Lecture de Platon*, ed. and trans. into French by J. Depuis (Paris, 1892; reprinted, Bruxelles, 1966), pp. 26-36.

able to find the true; without this kind of thought, nothing of truth is rightly known. This is the knowledge of those things which truly are; it is their full understanding and comprehension. He who spurns these, the paths of wisdom, does not rightly philosophize. Indeed, if philosophy is the love of wisdom, in spurning these, one has already shown contempt for philosophy.[9]

To this I think I should add that every force of a multitude, progressing from one point, moves on to limitless increases of growth. But a magnitude, beginning with a finite quantity, does not receive a new mode of being by division; its name includes the smallest sections of its body. This infinite and unlimited ability of nature in a multitude, philosophy spontaneously rejects. For nothing which is infinite is able to be assembled by a science or to be comprehended by the mind. But reason itself takes this matter of the infinite to itself; in these matters, reason is able to exercise the searching power of truth. It delegates the boundary of finite quality to the plurality of infinite multitude, and having rejected the aspect of interminable magnitude, it demands in a defined area a cognition of these things on its own behalf.

It stands to reason that whoever puts these matters aside has lost the whole teaching of philosophy. This, therefore, is the *quadrivium* by which we bring a superior mind from knowledge offered by the senses to the more certain things of the intellect. There are various steps and certain dimensions of progressing by which the mind is able to ascend so that by means of the eye of the mind, which (as Plato says)[10] is composed of many corporeal eyes and is of higher dignity than they, truth can be investigated and beheld. This eye, I say, submerged and surrounded by the corporal senses, is in turn illuminated by the disciplines of the *quadrivium*.

Which of these disciplines, then, is the first to be learned but that one

9. On the primary place of mathematical studies, see Iamblichus, *In Nicomachi Arithmeticam Introductio Liber*, ed. Hermengildus Pistelli (Leipzig, Teubner, 1894), p. 9; Theon, pp. 25-27.

10. Boethius summarizes Nichomachus' quotation from *Republic*, Sec. 527D. Nichomachus cites from memory or from a corrupt text since his version, below, differs from any other known version:

»You amuse me because you seem to fear that these are useless studies that I recommend; but that is very difficult, nay, impossible. For the eye of the soul, blinded and buried by other pursuits, is rekindled and aroused by these and these alone, and it is better that this be saved than the thousands of bodily eyes, for by it alone is the truth of the universe beheld.« D'ogge, pp. 186-187. Theon of Smyrna quotes the same passage, p. 7.

which holds the principal place and position of a mother to the rest?[11] This is artithmetic. It is prior to all not only because God the creator of the massive structure of the world considered this first discipline as the exemplar of his own thought and established all things in accord with it; or that through numbers of an assigned order all things exhibiting the logic of their maker found concord; but arithmetic is said to be first for this reason also, because whatever things are prior in nature, it is to these underlying elements that the posterior elements can be referred. Now if posterior things pass away, nothing concerning the status of the prior substance is disturbed--so, »animal« comes before »man«.[12] Now if you take away[13] »animal«, immediately also is the nature of »man« erased. If you take away »man«, »animal« does not disappear.

On the other hand, those things which are posterior infer prior things in themselves, and when these prior things are stated, they do not include in them anything of the posterior, as can be seen in that same term »man«. If you say »man«, you also say »animal«, because it is the same as man, since man is an animal. If you say »animal« you do not at the same time include the species of man, because »animal« is not the same as »man«.

The same thing is seen to occur in geometry and arithmetic. If you take away numbers, in what will consist the triangle, quadrangle, or whatever else is treated in geometry? All of those things are in the domain of number. If you were to remove the triangle and the quadrangle and all of geometry, still »three« and »four« and the terminology of the other numbers would not perish. Again, when I name some geometrical form, in that term the numbers are implicit. But when I say numbers, I have not implied any geometrical form.

The logical force of numbers is also prior to music, and this can especially be demonstrated because not only are numbers prior by their nature, since they consist of themselves and are thus prior to those things which must be referred to another in order to be, but also musical modulation itself is denoted by the names of numbers. The same relationship which we remarked in geometry can be found in music. The names diatessaron, dia-

11. See Plato, *Republic*, Sec. 522.
12. For the use of this argument, see Aristotle, *Topics*, Book 2, chap. 4.
13. Having shown to his satisfaction that number is metaphysically prior in creation, Boethius is now more concerned with logical priority. Hence here he means to »take away« or »abolish« in thought, logically. Nicomachus uses the Greek verb *συναναιρεῖθαι* with the same meaning, as does Aristotle in *Metaphysics* Sec. 1059B and Iamblichus, p. 10. (*τὰ δὲ συναναιροῦντα μὲν μὴ συναναιρούμενα δέ*)

pente, and diapason[14] are derived from the names of antecedent numerical terms. The proportion of their sounds is found only in these particular relationships and not in other number relationships. For the sound which is in a diapason harmony, the same sound is produced in the ratio of a number doubled. The interval of a diatessaron is found in an epitrita comparison; they call that harmony diapente which is joined by a hemiola interval. An epogdous in numbers is a tone in music. I cannot undertake to explain every consequence of this idea, as to how arithmetic is prior, but the rest of this work will demonstrate it without any doubt.

Arithmetic also precedes spherical and astronomical science insofar as these two remaining studies follow the third [geometry] naturally. In astronomy, »circles«, »a sphere«, »a center«, »concentric circles«, »the median« and »the axis« exist, all of which are the concern of the disciplineof geometry. For this reason, I want to demonstrate the anterior logical force of geometry. This is the case because in all things, movement naturally comes after rest; the static comes first. Thus, geometry understands the doctrine of immovable things while astronomy comprehends the science of mobile things. In astronomy, the very movement of the stars is celebrated in harmonic intervals. From this it follows that the power of music logically precedes the courses of the stars; and there is no doubt that arithmetic precedes astronomy since it is prior to music, which comes before astronomy. All the courses of the stars and all astronomic reasoning are established exclusively by the nature of numbers. Thus we connect rising with falling, thus we keep watch on the slowness and speed of wandering stars, thus we recognize the eclipses and multiplicities of lunar variations. Since, as it is obvious, the force of arithmetic is prior, we may take up the beginning of our exposition.

2. Concerning the substance of number.

From the beginning, all things whatever which have been created may be seen by the nature of things to be formed by reason of numbers.[15]

14. A diatessaron is a relation of 4:3, an interval of a fourth, or an epitrita; a diapente is a 3:2 relation, an interval of a fifth, or a hemiola; a diapason is an interval of an octave; an epogdous is the relation of 9:8 or a whole tone interval. For a definition of these terms, see Boethius, *De Institutione Musica*, Book 1, chap. 16-19.

Boethius begins his discussion of logical priority in the arts with arithmetic and geometry because logical priority is more obvious in the case of those disciplines, not because geometry follows arithmetic. He makes it amply clear earlier that music must follow arithmetic.

15. See Nicomachus, Book 1, chap.6.

Number was the principal exemplar in the mind of the creator. From it was derived the multiplicity of the four elements, from it were derived the changes of the seasons, from it the movement of the stars and the turning of the heavens. Since things are thus and since the status of all things is founded on the binding together of numbers, it is necessary that number in its own substance maintain itself evenly at all times, permanently, and that it not be composed of diverse elements. What substance would one join with number when the model of it itself holds all things together? It seems to have been composed of itself alone. Nothing can be considered as composed of similar parts or composed of things which are joined without reasonable proportion. Numbers are discrete of themselves and differ from every other substance and nature. But it becomes evident that number is composed of parts, not similar parts, nor of those things which adhere to each other without reasonable proportion. There are, therefore, first principles which join numbers together, which are in accord with its substance and which are always permanent. Nothing can be made from that which does not exist, and things from which something is made must be dissimilar but must possess the capacity of being combined. These then are the principles of which number consists: even and odd. These elements are disparate and contrary by a certain divine power, yet they come forth from one source and are joined into one composition and harmony.

3. The definition and division of number; the definition of even and odd.

First we must define what number is.[16] A number is a collection of unities, or a big mass of quantity issuing from unities.[17] Its first division therefore is into even and odd. Even is that which is able to be divided into

16. Nicomachus, Book 1, chap. 7. For another definition of number which is derived from Boethius, see Jordanus Nemorarius, *Arithmetica*, Book 1, Preface; see also Martianus Capella, *De Nuptiis Philologiae et Mercurii*, ed. A. Dick, revised by Jean Préaux (Stuttgart, Teubner, 1969), Sec. 743-49; 768-71; Theon, pp. 29-31.

17. Boethius, following Nicomachus, gives a three-fold definition of number:

1) »Limited Multitude«. According to this definition number is merely a species of the genus multitude with the differentia limitation. For a similar definition, see Artistotle, *Metaphysics*, Book V, chap. 13. Eudoxus used the same definition, according to Iamblichus (p.10).
2) »A Combination of Monads«. This definition is found in Theon (p.29) and, according to Iamblichus (p.10), is as old as Thales.
3) »A Flow of Number, composed of Monads«. As such, number is a stream, and it flows from monad. This definition has been attributed to Moderatus of Gades a Pythagorean. See Stobaeus, *Eclogae Physicae*, Book 1, chap. 2 and Robbins and Karpinski, pp. 114-5.

equal parts without one coming between the two parts. Odd is that which is unable to be divided into equal parts unless the aforesaid one should come between the parts. This kind of definition is common and well known.[18]

4. The definition of even and odd number according to Pythagoras.

Yet the definition of number is different according to the teaching of Pythagoras. An even number is that which at the same and single division is able to be divided into very large parts with small spaces or into a very small number of parts, with large spaces, according to the contrary properites of these two types.[19] An odd number is one to which this cannot happen but whose separation into two uneven parts is natural. Here is an example. If some given even number is divided, neither part is found to be larger than the other; there is nothing but a separation into halves. No quantity is smaller when a division of each of these halves into equal parts is made. In this way the even number 8 is divided into 4 and 4. There cannot be any other division which would divide this term into a smaller number of parts since there are no fewer parts than two. Now when a whole is divided in a three-fold division, the total space within each part is diminished, but the number of the divisions is increased. What was said about the contrary natures of the two types of numbers is relevant to this kind of situation. We have previously explained that a quantity grows into infinite pluralities, and that a space, which may otherwise be called a magnitude, can be diminished into infinitely small sections and that this occurs in contrary ways. This is the case in the division of an even number: when the space is maximum, the quantity is minimum.

5. Another definition of even and odd, according to a more ancient method.

According to a more ancient method, there is another definition of an even number. An even number is that which can be divided into two equal or two unequal parts, but in neither division is there an even number mixed with an odd number or an odd mixed with an even number, except-

18. On Even Number, see Jordanus, Book 7, prop. 2, 10 and 12; on Odd Number, Book 7, prop.3, 10, 16.

19. Here is described the contrariety of magnitude and quantity. Halves are the greatest possible parts of a term in magnitude, yet there is a smaller number of them in a whole than of any other fractionl part; see Iamblichus, pp. 11-13. For the »Lamboid Diagram« which illustrates this principle, see D'ooge, p. 191.

ing alone the principal even binary number which cannot be divided into two unequal parts because it consists only of two unities. This number is the first equality, two. What I am saying is this: if one takes an even number, it can be divided into two even parts, as ten is divided into fives. It can also be divided into unequal parts, as when the same ten is divided into 3 and 7. In this manner, when one part of a division is even, the other part is even; and if one part is odd, the other part, which is also odd, when added to it does not make a total of more or less than ten. When ten is divided into fives, or into three and seven, and these portions are divided further, only odd numbers result. If, however, this number or any other even number is divided into even parts, as when eight is divided into 4 and 4, or in the same manner into odd numbers, as when the same eight is divided into 5 and 3, this is the result: in the first division both parts are even, and in the second both are odd. If one part of a division is even, the other cannot be odd, and if one part is odd, the other cannot be even. An odd number is that which is divided into other odd numbers by means of any division; numbers always show both types and neither type of numbers is ever able to exist without the other, but one must be understood as even, the other as odd. If you divide seven into three and four, one part is even, the other is odd. This same condition is found to exist in all odd numbers, and it can never be otherwise in the division of an odd number. These are the two types which naturally make up the power and substance of number.

6. The definition of even and odd in terms of each other.

Now if these types must be defined in terms of each other, the odd number may be said to be that which differs from the even by a unity, either by increase or reduction. In the same way, an even number is one which differs from an odd number by a unity, either by increase or reduction. If you subtract one from an even number or add one to it, you make it odd; if you do the same thing to an odd number, an even number is created thereupon.

7. Concerning the primary nature of unity.

According to the natural arrangement of things, every number is half of the sum of the numbers which come before and after it.[20] And if the

20. Nichomachus, Book 1, chap. 8; see also Jordanus, Book 1, prop.2. The theory may be stated algebraically as

numbers beyond these, whose midpoint is the given number, are added together, then the given number would be half of their total. This is true of the number above that joined to the one below the previous lower number down to the end which is unity. So, if one takes the number 5, the numbers around it are, above, 4, below, 6. If these are joined, they add up to 10 whose half is 5. The numbers which are beyond these, that is beyond 6 and 4, namely 3 and 7, if joined, also equal twice the number 5. Also the numbers beyond these, if they are joined together, are double five. These are two and eight. If they are joined, they add up to 10, whose half, again, is five. The same thing happens with all numbers, and the process can be repeated to the terminal point of unity. Only unity does not have next to it two terms, and so unity is half of that number which is next to it and of it exclusively. Next to one, only two is naturally placed, and unity is evenly a half of two. For that reason it constitutes the primary unit of all numbers which are in the natural order and is rightly recognized as the generator of the total extended plurality of numbers.

8. The division of even number.

There are three types of even number. One is called even times even, another even times odd, and a third odd times even.[21] The contrary types seem to be in opposite places, and these are the even times even and even

$$n = \frac{(n-1) + (n+1)}{2}$$

Anyone discussing the history of ancient mathematics should heed Jacob Klein's warning that the use of modern mathematical symbols distorts the fundamental concepts of early mathematics (p.5). I have used such symbols sparingly and only when I felt they would clarify an otherwise obscure text. D'ooge and Dupuis may be consulted for a fuller use of such symbols. It should also be noted that Boethius uses Roman numerals (for which I substitute Arabic) or the written word for a numeral, apparently without a consistent reason for taking one over the other. I have followed his designation in each instance.

21. This classification is found in Euclid, *Elements*, Book 7, where he gives four kinds of numbers: even times even, even times odd, odd times even, and odd times odd. Since the fourth type is not an even number, it is omitted by Nicomachus and Boethius. See Iamblichus, Book 1, chap. 87 and Theon, pp. 41-43. Jordanus gives this classification in Book 7, prop. 29, 31, and 32. See also Nesselmann, *Die Algebra der Griechen* (Berlin, G. Reimer, 1842), p. 192 and T. L. Heath, *A History of Greek Mathematics* (Oxford, Clarendon Press, 1921), vol. I, p. 70. Theon defines even times even as:

a) That number produced by the multiplication of two numbers.

b) That number all of whose parts are even, down to unity.

c) That number which has none of its designations in terms of odd numbers.

times odd. The middle type which participates in both the other two is the number called odd times even.

9. Concerning the number even times even and its properties.

The even times even number is that which is able to be divided into two equal parts, and those parts again into two equal parts, and then into further equal parts. This is done until the division of parts arrives naturally at indivisible unity. So the number 64 has a half of 32, and this has a half of 16, and this a half of 8; from here the four, which is a double of the binary, is divided into equals. The binary is divided by the half of unity and unity is naturally singular and does not accept division. We can see that however many parts of this kind a number may have, the term even times even applies to each of them and the same is true of their sum. So it thus seems to me that this number is called even times even because all of its parts, both in name and quantity, are found to be even times even. How and in what name and quantity this number would have even parts, we shall say later. Their generation, however, is thus: whatever numbers you write in double proportions after one, these are always generated as even times even numbers. It is impossible that they be born otherwise than through this process of generation. This matter may be shown by means of a descriptive example. Let there be put down all the doubles after one: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512. If an infinite progression were made in this fashion you would find all the numbers in this progression. They are derived from one in double proportion and all of them are even times even numbers. Moreover, it is worthy of more than slight consideration that each of its parts is given the same name as all other parts of the series, namely even times even, and includes as great a total perfection of the quantity as is the number part of the quantity of even times even which it contains. It is such that the parts of a number correspond to each other so that however great one part is, the other has the same quantity, and however much that part is, it follows that it is necessary that the sum of that multitude be discovered within the former number. [...]