10.1090/spec/042/16

Descartes commanded the future from his study more than Napoleon from his throne.
— Oliver Wendell Holmes (1809–1894)

History of Mathematics: Why and How

André Weil

André Weil's life (1906–1998) spanned virtually the entire 20th century during which mathematics underwent a dramatic and exciting evolution and transformation. Part of this advancement is due to Weil himself.

He was born in Paris the son of Bernard Weil, a physician, and Selma Reinharz. Bernard belonged to a Jewish family from Alsace (which was part of Germany at the time of his birth) while Selma came from a middle class Austrian-Jewish family that had settled in Rostov-on-Don but eventually moved to Paris. A sister Simone was born in 1909; she had a very sad ending dying in Britain during the war in 1943. The two were close as children and remained so throughout Simone's short life. Trained in philosophy, she is now widely revered for her religious mysticism.

André was precocious and, in addition to his natural gifts for mathematics, he took a great interest in, and had an inherent talent for, languages. Among other things, he had a passion for Sanskrit and the Baghavad Gita, and for Greek poetry. He traveled extensively meeting many mathematicians in his journeys. He was awarded the degree DSc from Paris in 1928. He then spent two years at Aligarh University in India where he absorbed the culture of Buddhism. He was then on the faculty of the University of Strasbourg from 1933 until the outbreak of the Second World War. While in Strasbourg, he was one of the main forces in the establishment of "Bourbaki," a group of mathematicians determined to rewrite the whole of mathematics on a firm logical foundation much as in the style that Euclid achieved for geometry.

At this point in his life, there began an odyssey that did not come to an end until he eventually made his way to the U.S.A. in 1941. The source of the problem was his quasi-pacifism. From his own testimony, he was not a pacifist in the strict sense of the word but he had no intention or desire to participate in the armed services. His avowed reason is that in the First World War, the flower of French youth was decimated and this applied especially to young mathematicians. At the outbreak of WWII, French mathematics had not yet recovered from the catastrophe.

Weil went to Finland where he was hospitably received, and had hoped to make his way to the USA but that was not to be. He was required to return to France where he was imprisoned for not reporting for duty. He was released and eventually made his way, via Marseille, to the USA where he arrived in 1941. He taught at Haverford College and Lehigh University. In 1945 he went to São Paulo University in Brazil where he remained till 1947. He then went to the University of Chicago and the Institute for Advanced Study from which he retired in 1976.

In 1936 he married Eveline, who had recently been divorced from her first husband. She had a son Alain. The marriage was a happy one and the Weils had two daughters, Sylvie and Nicolette. Eveline's death in 1986 was a severe blow. Indeed in his autobiography, Weil describes his life as consisting of the interval from birth to his wife's death.

Weil was a very original thinker and left a large volume of important mathematics but, equally significantly, his creative mind opened new pathways that other mathematicians have successfully pursued.

He is said to have had a strong sense of humor and a sharp wit seasoned with doses of sarcasm directed at what he regarded as infelicitous behavior.

Weil received many honors including the noted Kyoto Prize as well as the Wolf Prize and the Steele Prize of the American Mathematical Society.

Editor's Preface

No one ever questions the pertinence of political history (Henry Ford may have been an exception!) or the history of art or music. Yet historians of mathematics feel the obligation to justify the relevance of this undertaking.

A renowned mathematician and mathematical historian—André Weil—rises to the challenge. Indeed at the time of the 1950 Congress (Cambridge, MA) Weil's book on the historical development of elliptic functions had already appeared and there was soon to appear his history of the theory of numbers entitled *Number Theory: An Approach through History from Hammurapi to Legendre.*

Many accounts of political history consist mainly of a recitation of events and dates and in many traditional history courses in schools, this is the primary content. More thoughtful historians however, endeavor to trace the evolution of political thought through the ages. Consider for example the concept of a democratic form of government. It is a deep concept and not one that would readily come to mind as a method of governing a political entity. Its development went through many stages and the thoughts of many writers, through history, have been distilled and amalgamated to propose a workable system that has functioned successfully in numerous contexts. The search for the origins of these ideas is a valuable adjunct to the ideas themselves.

So it is with the evolution of scientific ideas and, in particular, mathematical ones. A mathematical idea may undergo a metamorphosis over time. The development of the idea may not be "linear," i.e., mathematician A may have contributed a deep mathematical insight that is subsequently shrouded in mist and temporarily forgotten. A historical study may resuscitate the original idea and may shed light on its reincarnation. Weil cites several examples and suggests that many more might be revived if care is taken to delve into their history. Weil also surmises that the personality and life of a creative mathematician may infuse life into that person's creations. Why this should be so seems on the face of it unconvincing but Weil suggests that, for example, knowing that Euler lived in St. Petersburg and that he initially worked for the Russian Navy endows his work with a humanity not otherwise felt.

But there is an additional bonus that comes from exploring the historical genesis of a mathematical idea. This arises as follows: Often in learning mathematics, even at an elementary level, a concept or result is presented in its mature form. By contrast it is, at times, exciting to witness the evolution from conception to fruition. This writer can testify that one of the interesting events experienced was reading one of Euler's memoirs dealing with an important theorem. The original had a clarity of which the reader had been unaware. In a different direction, Weil gives as an example the invention of logarithms. The logarithmic function, in contemporary accounts, is presented as an isomorphism. It is illuminating to see that Napier viewed the logarithm as the relation between the motion of two points moving under different constraints.

What is surprising and noteworthy is that mathematical historians have been active throughout the centuries. The Duc de Montmort, who compiled a history of geometry, wrote to Nicholas Bernoulli as follows: "We have history of painting, of music, of medicine. A history of mathematics would be more interesting and useful.... it could be regarded as a history of the human spirit for it is in this science more than in others that man makes known the excellence of the intelligence that God has given him." Weil cites several historians including a pupil of Aristotle named Eudemus who belonged to the school of the peripatetics. Another of Aristotle's pupils, Theophrastus wrote a history of arithmetic, geometry and astronomy. Histories have appeared through the ages and one of the more notable accounts was given by Jean Étienne Montucla in 1756. In the first half of the 20th century we witness a monumental work by Moritz Cantor and in the late decades, a flood of books on history of mathematics and related topics—a testimony to the growing interest in the origins and historical elucidations of mathematical ideas. And a tribute to Weil's influence and predictions on the significance of historical studies.

My first point will be an obvious one. In contrast with some sciences whose whole history consists of the personal recollections of a few of our contemporaries, mathematics not only has a history but it has a long one, which has been written about at least since Eudemos (a pupil of Aristotle). Thus the question "Why?" is perhaps superfluous, or would be better formulated as "For whom?".

For whom does one write general history? for the educated layman, as Herodotus did? for statesmen and philosophers, as Thucydides? for one's fellow-historians, as is mostly done nowadays? What is the right audience for the art-historian? his colleagues, or the art-loving public, or the artists (who seem to have little use for him)? What about the history of music? Does it concern chiefly music-lovers, or composers, or performing artists, or cultural historians, or is it a wholly independent discipline whose appreciation is confined to its own practitioners? Similar questions have been hotly debated for many years among eminent historians of mathematics, Moritz Cantor, Gustav Eneström, Paul Tannery. Already Leibniz had something to say about it, as about most other topics:

"Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples." [1]

That mankind should be spurred on by the prospect of eternal fame to ever higher achievements is of course a classical theme, inherited from antiquity; we seem to have become less sensitive to it than our forefathers were, although it has perhaps not quite spent its force. As to the latter part of Leibniz' statement, its purport is clear. He wanted the historian of science to write in the first place for creative or would-be creative scientists. This was the audience he had in mind while writing in retrospect about his "most noble invention" of the calculus.

On the other hand, as Moritz Cantor observed, one may, in dealing with mathematical history, regard it as an auxiliary discipline, meant for providing the true historian with reliable catalogues of mathematical facts, arranged according to times, countries, subject-matters and authors. It is then a portion, and not a very significant one, of the history of techniques and crafts, and it is fair to look upon it entirely from the outside. The historian of the XIXth century needs some knowledge of the progress made by the railway engine; for this he has to depend upon specialists, but he does not care how the engine works, nor about the gigantic intellectual effort that went into the creation of thermodynamics. Similarly, the development of nautical tables and other aids to navigation is of no little importance for the historian of XVIIth century England, but the part taken in it by Newton will provide him at best with a footnote; Newton as keeper of the Mint, or perhaps as the uncle of a great nobleman's mistress, is closer to his interests than Newton the mathematician.

From another point of view, mathematicians may occasionally provide the cultural historian with a kind of "tracer" for investigating the interaction between various cultures. With this we come closer to matters of genuine interest to us mathematicians; but even here our attitudes differ widely from those of professional historians. To them a Roman coin, found somewhere in India, has a definite significance; hardly so a mathematical theory.

This is not to say that a theorem may not have been rediscovered time and again, even in quite different cultural environments. Some power-series expansions seem to have been discovered independently in India, in Japan and in Europe. Methods for the solution of Pell's equation were expounded in India by Bhaskara in the XIIth century, and then again, following a challenge from Fermat, by Wallis and Brouncker in 1657. One can even adduce arguments for the view that similar methods may have been known to the Greeks, perhaps to Archimedes himself; as Tannery suggested, the Indian solution could then be of Greek origin; so far this must remain an idle speculation. Certainly no one would suggest a connection between Bhaskara and our XVIIth century authors.

On the other hand, when quadratic equations, solved algebraically in cuneiform texts, surface again in Euclid, dressed up in geometric garb without any geometric motivation at all, the mathematician will find it appropriate to describe the latter treatment as "geometric algebra" and will be inclined to assume some connection with Babylon, even in the absence of any concrete "historical" evidence. No one asks for documents to testify to the common origin of Greek, Russian and Sanskrit, or objects to their designation as indo-european languages.

Now, leaving the views and wishes of laymen and of specialists of other disciplines, it is time to come back to Leibniz and consider the value of mathematical history, both intrinsically and from our own selfish viewpoint as mathematicians. Deviating only slightly from Leibniz, we may say that its first use for us is to put or to keep before our eyes "illustrious examples" of first-rate mathematical work.

Does that make historians necessary? Perhaps not. Eisenstein fell in love with mathematics at an early age by reading Euler and Lagrange; no historian told him to do so or helped him to read them. But in his days mathematics was progressing at a less hectic pace than now. No doubt a young man can now seek models and inspiration in the work of his contemporaries; but this will soon prove to be a severe limitation. On the other hand, if he wishes to go much further back, he may find himself in need of some guidance; it is the function of the historian, or at any rate of the mathematician with a sense for history, to provide it.

The historian can help in still another way. We all know by experience how much is to be gained through personal acquaintance when we wish to study contemporary work; our meetings and congresses have hardly any other purpose. The life of the great mathematicians of the past may often have been dull and unexciting, or may seem so to the layman; to us their biographies are of no small value in bringing alive the men and their environment as well as their writings. What mathematician would not like to know more about Archimedes than the part he is supposed to have taken in the defense of Syracuse? Would our understanding of Euler's number theory be quite the same if we merely had his publications at our disposal? Is not the story infinitely more interesting when we read about his settling down in Russia, exchanging letters with Goldbach, getting almost accidentally acquainted with the works of Fermat, then, much later in life, starting a correspondence with Lagrange on number theory and elliptic integrals? Should we not be pleased that, through his letters, such a man has come to belong to our close acquaintance?

So far, however, I have merely scratched the surface of my theme. Leibniz recommended the study of "illustrious examples," not just for the sake of esthetic enjoyment, but chiefly so that "the art of discovery be promoted." At this point one has to make clear the distinction, in scientific matters, between tactics and strategy.

By tactics I understand the day-to-day handling of the tools at the disposal of the scientist or scholar at a given moment; this is best learnt from a competent teacher and the study of contemporary work. For the mathematician it may include the use of differential calculus at one time, of homological algebra at another. For the historian of mathematics, tactics have much in common with those of the general historian. He must seek his documentation at its source, or as close to it as practicable; second-hand information is of small value. In some areas of research one must learn to hunt for and read manuscripts; in others one may be content with published texts, but then the question of their reliability or lack of it must always be kept in mind. An indispensable requirement is an adequate knowledge of the language of the sources; it is a basic and sound principle of all historical research that a translation can never replace the original when the latter is available. Luckily the history of Western mathematics after the XVth century seldom requires any linguistic knowledge besides Latin and the modern Western European languages; for many purposes French, German and sometimes English might even be enough.

In contrast with this, strategy means the art of recognizing the main problems, attacking them at their weak points, setting up future lines of advance. Mathematical strategy is concerned with long-range objectives; it requires a deep understanding of broad trends and of the evolution of ideas over long periods. This is almost indistinguishable from what Gustav Eneström used to describe as the main object of mathematical history, viz., "the mathematical ideas, considered historically" [2], or, as Paul Tannery put it, "the filiation of ideas and the concatenation of discoveries." [3] There we have the core of the discipline we are discussing, and it is a fortunate fact that the aspect towards which, according to Eneström and Tannery, the mathematical historian has chiefly to direct his attention is also the one of greatest value for any mathematician who wants to look beyond the everyday practice of his craft.

The conclusion we have reached has little substance, to be sure, unless we agree about what is and what is not a mathematical idea. As to this, the mathematician is hardly inclined to consult outsiders. In the words of Housman (when asked to define poetry), he may not be able to define what is a mathematical idea, but he likes to think that when he smells one he knows it. He is not likely to see one, for instance, in Aristotle's speculations about the infinite, nor in those of a number of medieval thinkers on the same subject, even though some of them were rather more interested in mathematics than Aristotle ever was; the infinite became a mathematical idea after Cantor defined equipotent sets and proved some theorems about them. The views of Greek philosophers about the infinite may be of great interest as such; but are we really to believe that they had great influence on the work of Greek mathematicians? Because of them, we are told, Euclid had to refrain from saying that there are infinitely many primes, and had to express that fact differently. How is it then that, a few pages later, he stated that "there exist infinitely many lines" incommensurable with a given one? Some universities have established chairs for "the history and philosophy of mathematics": it is hard for me to imagine what those two subjects can have in common.

Not so clear-cut is the question where "common notions" (to use Euclid's phrase) end and where mathematics begins. The formula for the sum of the first n integers, closely related as it is to the "Pythagorean" concept of triangular numbers, surely deserves to be called a mathematical idea; but what should we say about elementary commercial arithmetic, as it appears in ever so many textbooks from antiquity down to Euler's potboiler on the same subject? The concept of a regular icosahedron belongs distinctly to mathematics; shall we say the same about the concept of a cube, that of a rectangle, or that of a circle (which is perhaps not to be separated from the invention of the wheel)? Here we have a twilight zone between cultural and mathematical history;

it does not matter much where one draws the borderline. All the mathematician can say is that his interest tends to falter, the nearer he comes to crossing it.

However that may be, once we have agreed that mathematical ideas are the true object of mathematical history, some useful consequences can be drawn; one has been formulated by Tannery as follows (*loc. cit.,* (footnote 3), p. 164). There is no doubt at all, he says, that a scientist can possess or acquire all the qualities needed to do excellent work on the history of his science; the greater his talent as a scientist, the better his historical work is likely to be. As examples, he mentions Châsles for geometry; also Laplace for astronomy, Berthelot for chemistry; perhaps he was also thinking of his friend Zeuthen. He might well have quoted Jacobi, if Jacobi had lived to publish his historical work. [4]

But examples are hardly necessary. Indeed it is obvious that the ability to recognize mathematical ideas in obscure or inchoate form, and to trace them under the many disguises which they are apt to assume before coming out in full daylight, is most likely to be coupled with a better than average mathematical talent. More than that, it is an essential component of such talent, since in large part the art of discovery consists in getting a firm grasp on the vague ideas which are "in the air," some of them flying all around us, some (to quote Plato) floating around in our own minds.

How much mathematical knowledge should one possess in order to deal with mathematical history? According to some, little more is required than what was known to the authors one plans to write about; [5] some go so far as to say that the less one knows, the better one is prepared to read those authors with an open mind and avoid anachronisms. Actually the opposite is true. An understanding in depth of the mathematics of any given period is hardly ever to be achieved without knowledge extending far beyond its ostensible subject matter. More often than not, what makes it interesting is precisely the early occurrence of concepts and methods destined to emerge only later into the conscious mind of mathematicians; the historian's task is to disengage them and trace their influence or lack of influence on subsequent developments. Anachronism consists in attributing to an author such conscious knowledge as he never possessed; there is a vast difference between recognizing Archimedes as a forerunner of integral and differential calculus, whose influence on the founders of the calculus can hardly be overestimated, and fancying to see in him, as has sometimes been done, an early practitioner of the calculus. On the other hand, there is no anachronism in seeing in Desargues the founder of the projective geometry of conic sections; but the historian has to point out that his work, and Pascal's, soon fell into the deepest oblivion, from which it could only be rescued after Poncelet and Chasles had independently rediscovered the whole subject.

Similarly, consider the following assertion: logarithms establish an isomorphism between the multiplicative semigroup of numbers between 0 and 1 and the additive semigroup of positive real numbers. This could have made no sense until comparatively recently. If, however, we leave the words aside and look at the facts behind that statement, there is no doubt that they were well understood by Neper [Napier] when he invented logarithms, except that his concept of real numbers was not as clear as ours; this is why he had to appeal to kinematic concepts in order to clarify his meaning, just as Archimedes had done, for rather similar reasons, in his definition of the spiral. [6] Let us go further back; the fact that the theory of the ratios of magnitudes and of the ratios of integers, as developed by Euclid in Books V and VII of his Elements, is to be regarded as an early chapter of group-theory is put beyond doubt by the phrase "double ratio" used by him for what we call the square of a ratio. Historically it is quite plausible that musical theory supplied the original motivation for the Greek theory of the group of ratios of integers, in sharp contrast with the purely additive treatment of fractions in Egypt; if so, we have there an early example of the mutual interaction between pure and applied mathematics. Anyway, it is impossible for us to analyze properly the contents of Books V and VII of Euclid without the concept of group and even that of groups with operators, since the ratios of magnitudes are treated as a multiplicative group operating on the additive group of the magnitudes themselves. [7] Once that point of view is adopted, those books of Euclid lose their mysterious character, and it becomes easy to follow the line which leads directly from them to Oresme and Chuquet, then to Neper and logarithms. In doing so, we are of course not attributing the group concept to any of these authors; no more should one attribute it to Lagrange, even when he was doing what we now call Galois theory. On the other hand, while Gauss had not the word, he certainly had the clear concept of a finite commutative group, and had been well prepared for it by his study of Euler's number-theory.

Let me quote a few more examples. Fermat's statements indicate that he was in possession of the theory of the quadratic forms $X^2 + nY^2$ for

$n = 1, 2, 3$, using proofs by "infinite descent." He did not record those proofs; but eventually Euler developed that theory, also using infinite descent, so that we may assume that Fermat's proofs did not differ much from Euler's. Why does infinite descent succeed in those cases? This is easily explained by the historian who knows that the corresponding quadratic fields have an Euclidean algorithm; the latter, transcribed into the language and notations of Fermat and Euler, gives precisely their proofs by infinite descent, just as Hurwitz' proof for the arithmetic of quaternions, similarly transcribed, gives Euler's proof (which possibly was also Fermat's) for the representation of integers by sums of 4 squares.

Take again Leibniz' notation $\int ydx$ in the calculus. He insisted repeatedly on its invariant character, first in his correspondence with Tschirnhaus (who showed no understanding for it), then in the *Acta Eruditorum* of 1686; he even had a word for it ("*universalitas*"). Historians have hotly disputed when, or whether, Leibniz discovered the comparatively less important result that, in some textbooks, goes by the name of "the fundamental theorem of the calculus." But the importance of Leibniz' discovery of the invariance of the notation ydx could hardly have been properly appreciated before Elie Cartan introduced the calculus of exterior differential forms and showed the invariance of the notation $ydx_1 \ldots dx_m$, not only under changes of the independent variables (or of local coordinates), but even under "pull-back." [8]

Consider now the debate that arose between Descartes and Fermat about tangents. Descartes, having decided, once and for all, that only algebraic curves were a fit subject for geometers, invented a method for finding their tangents, based upon the idea that a variable curve, intersecting a given one C at a point P, becomes tangent to C at P when the equation for their intersections acquires a double root corresponding to P. Soon Fermat, having found the tangent to the cycloid by an infinitesimal method, challenged Descartes to do the same by his own method. Of course he could not do that; being the man he was, he found the answer (*Oeuvres,* II, p. 308), gave a proof for it ("quite short and quite simple") by using the instantaneous center of rotation which he invented for the occasion, and added that he could have supplied another proof "more to his taste and more geometrical" which he omitted "to save himself the trouble of writing it out"; anyway, he said, "such lines are mechanical" and he had excluded them from geometry. This, of course, was the point that Fermat was trying to make; he knew, as well as

Descartes, what an algebraic curve was, but to restrict geometry to those curves was quite alien to his way of thinking and to that of most geometers in the XVIIth century.

Gaining insight into a great mathematician's character and into his weaknesses is an innocent pleasure that even serious historians need not deny themselves. But what else can one conclude from that episode? Very little, as long as the distinction between differential and algebraic geometry has not been clarified. Fermat's method belonged to the former; it depended upon the first terms of a local power series expansion; it provided the starting point for all subsequent developments in differential geometry and differential calculus. On the other hand, Descartes' method belongs to algebraic geometry, but, being restricted to it, it remained a curiosity until the need arose for methods valid over quite arbitrary ground-fields. Thus the point at issue could not be and was not properly perceived until abstract algebraic geometry gave it its full meaning.

There is still another reason why the craft of mathematical history can best be practiced by those of us who are or have been active mathematicians or at least who are in close contact with active mathematicians; there are various types of misunderstandings of not infrequent occurrence from which our own experience can help preserve us. We know only too well, for instance, that one should not invariably assume a mathematician to be fully aware of the work of his predecessors, even when he includes it among his references; which one of us has read all the books he has listed in the bibliographies of his own writings? We know that mathematicians are seldom influenced in their work by philosophical considerations, even when they profess to take them seriously; we know that they have their own way of dealing with foundational matters by an alternation between possibly reckless disregard and the most painful critical attention. Above all, we have learnt the difference between original thinking and the kind of routine reasoning which a mathematician often feels he has to spin out for the record in order to satisfy his peers, or perhaps only to satisfy himself. A tediously laborious proof may be a sign that the writer has been less than felicitous in expressing himself; but more often than not, as we know, it indicates that he has been laboring under limitations which prevented him from translating directly into words or formulas some very simple ideas. Innumerable instances can be given of this, ranging from Greek geometry (which perhaps was at last suffocated by such limitations) down to the so-called epsilontic and down to Nicolas Bourbaki, who even once considered using a special sign in the margin to warn the reader about proofs of that kind. One important task of the serious historian of mathematics, and sometimes one of the hardest, is precisely to sift such routine from what is truly new in the work of the great mathematicians of the past.

Of course mathematical talent and mathematical experience are not enough for qualifying as a mathematical historian. To quote Tannery again (*loc. cit.* (footnote 3), p. 165), "what is needed above all is a taste for history; one has to develop a historical sense." In other words, a quality of intellectual sympathy is required, embracing past epochs as well as our own. Even quite distinguished mathematicians may lack it altogether; each one of us could perhaps name a few who resolutely refuse to be acquainted with any work other than their own. It is also necessary not to yield to the temptation (a natural one to the mathematician) of concentrating upon the greatest among past mathematicians and neglecting work of only subsidiary value. Even from the point of view of esthetic enjoyment one stands to lose a great deal by such an attitude, as every art-lover knows; historically it can be fatal, since genius seldom thrives in the absence of a suitable environment, and some familiarity with the latter is an essential prerequisite for a proper understanding and appreciation of the former. Even the textbooks in use at every stage of mathematical development should be carefully examined in order to find out, whenever possible, what was and what was not common knowledge at a given time.

Notations, too, have their value. Even when they are seemingly of no importance, they may provide useful pointers for the historian; for instance, when he finds that for many years, and even now, the letter K has been used to denote fields, and German letters to denote ideals, it is part of his task to explain why. On the other hand, it has often happened that notations have been inseparable from major theoretical advances. Such was the case with the slow development of the algebraic notation, finally brought to completion at the hands of Viète and Descartes. Such was the case again with the highly individual creation of the notations for the calculus by Leibniz (perhaps the greatest master of symbolic language that ever was); as we have seen, they embodied Leibniz' discoveries so successfully that later historians, deceived by the simplicity of the notation, have failed to notice some of the discoveries.

Thus the historian has his own tasks, even though they overlap those of the mathematician and may at times coincide with them. Thus, in the

XVIIth century, it happened that some of the best mathematicians, in the absence of immediate predecessors in any field of mathematics except algebra, had much work to do which in our view would fall to the lot of the historian—editing, publishing, reconstructing the work of the Greeks, of Archimedes, Apollonios, Pappos, Diophantos. Even now the historian and the mathematician will not infrequently find themselves on common ground when studying the production of the XIXth and XXth centuries, not to mention anything of more ancient vintage. From my own experience I can testify about the value of suggestions found in Gauss and in Eisenstein. Kummer's congruences for Bernoulli numbers, after being regarded as little more than a curiosity for many years, have found a new life in the theory of p-adic L-functions, and Fermat's ideas on the use of the infinite descent in the study of Diophantine equations of genus I have proved their worth in contemporary work on the same subject.

What, then, separates the historian from the mathematician when both are studying the work of the past? Partly, no doubt, their techniques, or, as I proposed to put it, their tactics; but chiefly, perhaps, their attitudes and motivations. The historian tends to direct his attention to a more distant past and to a greater variety of cultures; in such studies, the mathematician may find little profit other than the esthetic satisfaction to be derived from them and the pleasures of vicarious discovery. The mathematician tends to do his reading with a purpose, or at least with the hope that some fruitful suggestion will emerge from it. Here we may quote the words of Jacobi in his younger days about a book he had just been reading: "Until now," he said, "whenever I have studied a work of some value, it has stimulated me to original thoughts; this time I have come out quite empty-handed." As noted by Dirichlet, from whom I have borrowed this quotation, it is ironical that the book in question was no other than Legendre's *Exercices de calcul intégral*, containing work on elliptic integrals which soon was to provide the inspiration for Jacobi's greatest discoveries; but those words are typical. The mathematician does his reading mostly in order to be stimulated to original (or, I may add, sometimes not so original) thoughts; there is no unfairness, I think, in saying that his purpose is more directly utilitarian than the historian's. Nevertheless, the essential business of both is to deal with mathematical ideas, those of the past, those of the present, and, when they can, those of the future. Both can find invaluable training and enlightenment in each other's work. Thus my original question "Why mathematical history?" finally reduces itself to the question "Why mathematics?", which fortunately I do not feel called upon to answer.

Endnotes

[1] *"Utilissimum est cognosci veras in inventionum memorabilium origines, praesertim earum, quae non casu, sed vi meditandi innotuere. Id enim non eo tantum prodest, ut Historia literaria suum cuique tribuat et alii ad pares laudes invitentur, sed etiam ut augeatur ars inveniendi, cognita methodo illustribus exemplis. Inter nobiliora hujus temporis inventa habetur novum Analyseos Mathematicae genus, Calculi differentialis nomine notum...* (*Math. Schr.,* ed. C. I. Gerhardt, t. V, p. 392).

[2] *Die mathematischen Ideen in historischer Behandlung* (*Bibl. Math.* 2 (1901), p. 1)

[3] *La filiation des idées et l'enchaînement des découvertes* (P. Tannery, *Oeuvres,* vol. X, p. 166)

[4] Jacobi, as a student, had hesitated between classical philology and mathematics; he always retained a deep interest in Greek mathematics and mathematical history; extracts from his writings on this subject have been published by Koenigsberger in his biography of Jacobi (incidentally, a good model for a mathematically oriented biography of a great mathematician): see L. Koenigsberger, *Carl Gustav Jacob Jacobi,* Teubner, 1904, pp. 385–395 and 413–414.

[5] Such seems to have been Loria's view: "Per comprendere e giudicare gli scritti appartenenti alle età passate, basta di essere esperto in quelle parti delle scienze che trattano dei numeri e delle figure e che si considerano attualmente come parte della cultura generale dell'uomo civile" (G. Loria, *Guida allo Studio della Storia delle Matematiche,* U. Hoepli, Milano, 1946, p. 271).

[6] Cf. N. Bourbaki, *Eléments d'histoire des mathématiques*, Hermann, 1966, pp. 167–168 and 174; that collection of historical essays, extracted from the same author's *Eléments de mathématique* under a misleading title, will be quoted henceforth as NB.

[7] Whether or not Euclid believed the group of ratios of magnitudes to be independent of the kind of magnitudes under study is still a moot point; cf. O. Becker, *Quellen u. Studien* 2 (1933), pp. 369–387.

[8] Cf. NB, p. 208, and A.Weil, *Bull. Amer. Math. Soc.* 81 (1975), 683.