Pappus of Alexandria
The Collection
Book 7
Which contains lemmas of the *Domain of Analysis*

(1) That which is called the *Domain of Analysis*, my son Hermodorus, is, taken as a whole, a special resource that was prepared, after the composition of the *Common Elements*, for those who want to acquire a power in geometry that is capable of solving problems set to them; and it is useful for this alone. It was written by three men: Euclid the Elementarist, Apollonius of Perge, and Aristaeus the elder, and its approach is by analysis and synthesis.

Now, analysis is the path from what one is seeking, as if it were established, by way of its consequences, to something that is established by synthesis. That is to say, in analysis we assume what is sought as if it has been achieved, and look for the thing from which it follows, and again what comes before that, until by regressing in this way we come upon some one of the things that are already known, or that occupy the rank of a first principle. We call this kind of method 'analysis', as if to say *anapalin lysis* (reduction backward). In synthesis, by reversal, we assume what was obtained last in the analysis to have been achieved already, and, setting now in natural order, as precedents, what before were following, and fitting them to each other, we attain the end of the construction of what was sought. This is what we call 'synthesis'.

(2) There are two kinds of analysis: one of them seeks after truth, and is called 'theorematic'; while the other tries to find what was demanded, and is called 'problematic'. In the case of the theorematic kind, we assume what is sought as a fact and true, then, advancing through its consequences, as if they are true facts according to the hypothesis, to something established, if this thing that has been established is a truth, then that which was sought will also be true, and its proof the reverse of

ΠΑΠΠΟΥ ΑΛΕΞΑΝΔΡΕΩΣ ΣΥΝΑΓΩΓΗΣ Ζ´.
ΠΕΡΙΕΧΕΙ ΔΕ ΛΗΜΜΑΤΑ ΤΟΥ ΑΝΑΛΥΟΜΕΝΟΥ. 634

(1) ὁ καλούμενος Ἀναλυόμενος, Ἑρμόδωρε τέκνον, κατὰ σύλληψιν ἰδία τίς ἐστιν ὕλη παρασκευασμένη μετὰ τὴν τῶν κοινῶν στοιχείων ποίησιν τοῖς βουλομένοις ἀναλαμβάνειν ἐν γραμμαῖς δύναμιν εὑρετικὴν τῶν προτεινομένων αὐτοῖς προβλημάτων, καὶ εἰς τοῦτο μόνον χρησίμη καθεστῶσα. γέγραπται δὲ ὑπὸ τριῶν ἀνδρῶν, Εὐκλείδου τε τοῦ στοιχειωτοῦ καὶ Ἀπολλωνίου τοῦ Περγαίου καὶ Ἀρισταίου τοῦ πρεσβυτέρου, κατὰ ἀνάλυσιν καὶ σύνθεσιν ἔχουσα τὴν ἔφοδον. ἀνάλυσις τοίνυν ἐστὶν ὁδὸς ἀπὸ τοῦ ζητουμένου, ὡς ὁμολογουμένου, διὰ τῶν ἑξῆς ἀκολούθων, ἐπί τι ὁμολογούμενον συνθέσει. ἐν μὲν γὰρ τῆι ἀναλύσει, τὸ ζητούμενον ὡς γεγονὸς ὑποθέμενοι τὸ ἐξ οὗ [τοῦ] τοῦτο συμβαίνει σκοπούμεθα, καὶ πάλιν ἐκείνου τὸ προηγούμενον, ἕως ἂν οὕτως ἀναποδίζοντες καταντήσωμεν εἴς τι τῶν ἤδη γνωριζομένων ἢ τάξιν ἀρχῆς ἐχόντων. καὶ τὴν τοιαύτην ἔφοδον ἀνάλυσιν καλοῦμεν οἷον ἀνάπαλιν λύσιν. ἐν δὲ τῆι συνθέσει ἐξ ὑποστροφῆς τὸ ἐν τῆι ἀναλύσει καταληφθὲν ὕστατον ὑποστησάμενοι γεγονὸς ἤδη, καὶ τὰ ἑπόμενα ἐκεῖ, ἐνταῦθα προηγούμενα κατὰ φύσιν τάξαντες καὶ ἀλλήλοις ἐπισυνθέντες, εἰς τέλος ἀφικνούμεθα τῆς τοῦ ζητουμένου κατασκευῆς· καὶ τοῦτο καλοῦμεν σύνθεσιν.

(2) διττὸν δ' ἐστὶν ἀναλύσεως γένος· τὸ μὲν γὰρ ζητητικὸν τἀληθοῦς, ὃ καλεῖται θεωρητικόν, τὸ δὲ ποριστικὸν τοῦ προταθέντος [λέγειν] ὃ καλεῖται
προβληματικόν. ἐπὶ μὲν οὖν τοῦ θεωρητικοῦ γένους τὸ 636
ζητούμενον ὡς ὂν ὑποθέμενοι καὶ ὡς ἀληθές, εἶτα διὰ τῶν ἑξῆς ἀκολούθων ὡς ἀληθῶν καὶ ὡς ἔστιν καθ' ὑπόθεσιν προελθόντες ἐπί τι ὁμολογούμενον, ἐὰν μὲν ἀληθὲς ἦι ἐκεῖνο τὸ ὁμολογούμενον, ἀληθὲς ἔσται καὶ τὸ ζητούμενον καὶ ἡ

‖ 13 ante συνθέσει add ἐν Greg | γὰρ om Greg ‖ 14 τοῦ (in fine versus A) del Greg ‖ 18 τῆι om Ge ‖ 20 ἑπόμενα τὰ transp Hu ‖ 21 ἐνταῦθα secl. Hu ‖ 24 γὰρ om Ha ‖ 26 προτεθέντος Greg | λέγειν secl. Hu ‖ 29 ἀληθῶν Ha ἀληθῶς A | ἔστιν] ὄντων Hu app the analysis; but if we should meet with something established to be false, then the thing that was sought too will be false. In the case of the problematic kind, we assume the proposition as something we know, then, proceeding through its consequences, as if true, to something established, if the established thing is possible and obtainable, which is what mathematicians call 'given', the required thing will also be possible, and again the proof will be the reverse of the analysis; but should we meet with something established to be impossible, then the problem too will be impossible. Diorism is the preliminary distinction of when, how, and in how many ways the problem will be possible. So much, then, concerning analysis and synthesis.

(3) The order of the books of the *Domain of Analysis* alluded to above is this: Euclid, *Data,* one book; Apollonius, *Cutting off of a Ratio,* two; *Cutting off of an Area,* two; <*Determinate Section*>, two; *Tangencies,* two; Euclid, *Porisms,* three; Apollonius, *Neuses,* two; by the same, *Plane Loci,* two; *Conics,* eight; Aristaeus, *Solid Loci,* five; Euclid, *Loci on Surfaces,* two; Eratosthenes, *On Means* [two]. These make up 32 books. I have set out epitomes of them, as far as the *Conics* of Apollonius, for you to study, with the number of the dispositions and diorisms and cases in each book, as well as the lemmas that are wanted in them, and there is nothing wanting for the working through of the books, I believe, that have I left out.

(4) (The Data.)

The first book, which is the *Data,* contains ninety theorems in all. The first twenty-three diagrams are all about magnitudes. The twenty-fourth is on proportional lines that are not given in position. The fourteen next to these are on lines given in position. The next <ten> are on triangles given in shape without position. The next seven are on arbitrary rectilineal areas given in shape without position. The next six are on parallelograms and ἀπόδειξις ἀντίστροφος τῆι ἀναλύσει· ἐὰν δὲ ψεύδει ὁμολογουμένωι ἐντύχωμεν, ψεῦδος ἔσται καὶ τὸ ζητούμενον. ἐπὶ δὲ τοῦ προβληματικοῦ γένους τὸ προταθὲν ὡς γνωσθὲν ὑποθέμενοι, εἶτα διὰ τῶν ἑξῆς ἀκολούθων ὡς ἀληθῶν προελθόντες ἐπί τι ὁμολογούμενον, ἐὰν μὲν τὸ ὁμολογού|μενον δυνατὸν ἦι καὶ ποριστόν, ὃ καλοῦσιν οἱ ἀπὸ |119 τῶν μαθημάτων δοθέν, δυνατὸν ἔσται καὶ τὸ προταθέν, καὶ πάλιν ἡ ἀπόδειξις ἀντίστροφος τῆι ἀναλύσει· ἐὰν δὲ ἀδυνάτωι ὁμολογουμένωι ἐντύχωμεν ἀδύνατον ἔσται καὶ τὸ πρόβλημα. διορισμὸς δέ ἐστιν προδιαστολὴ τοῦ πότε καὶ πῶς καὶ ποσαχῶς δυνατὸν ἔσται [καὶ] τὸ πρόβλημα. τοσαῦτα μὲν οὖν περὶ ἀναλύσεως καὶ συνθέσεως.

(3) τῶν δὲ προειρημένων τοῦ Ἀναλυομένου βιβλίων ἡ τάξις ἐστὶν τοιαύτη. Εὐκλείδου Δεδομένων βιβλίον ᾱ, Ἀπολλωνίου Λόγου Ἀποτομῆς β̄, Χωρίου Ἀποτομῆς β̄, <Διωρισμένης Τομῆς> δύο, Ἐπαφῶν δύο, Εὐκλείδου Πορισμάτων τρία, Ἀπολλωνίου Νεύσεων δύο, τοῦ αὐτοῦ Τόπων Ἐπιπέδων δύο, Κωνικῶν η̄, Ἀρισταίου Τόπων Στερεῶν πέντε, Εὐκλείδου Τόπων πρὸς Ἐπιφανείαι δύο, Ἐρατοσθένους Περὶ Μεσοτήτων [δύο]. γίνεται βιβλία λβ̄, ὧν τὰς περιοχὰς μέχρι τῶν Ἀπολλωνίου Κωνικῶν ἐξεθέμην σοι πρὸς ἐπίσκεψιν, καὶ τὸ πλῆθος τῶν τόπων καὶ τῶν διορισμῶν καὶ τῶν πτώσεων καθ' ἕκαστον βιβλίον, ἀλλὰ καὶ τὰ λήμματα τὰ ζητούμενα, καὶ οὐδεμίαν ἐν τῆι πραγματείαι τῶν βιβλίων καταλέλοιπα ζήτησιν, ὡς ἐνόμιζον.

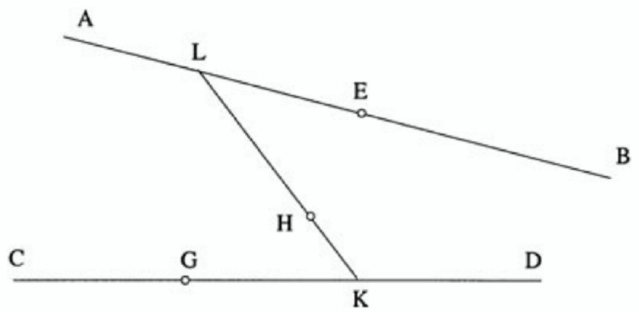
(4) περιέχει δὲ τὸ πρῶτον βιβλίον, ὅπερ ἐστὶν τῶν 638 Δεδομένων, ἅπαντα θεωρήματα ἐνενήκοντα. ὧν πρῶτα μὲν καθόλου ἐπὶ μεγεθῶν διαγράμματα κγ̄. τὸ δὲ δʹ καὶ [τὸ] κʹ ἐν εὐθείαις ἐστὶν ἀνάλογον ἄνευ θέσεως. τὰ δὲ ἑξῆς τούτοις ιδ̄ ἐν εὐθείαις ἐστὶν θέσει δεδομέναις. τὰ δὲ τούτοις ἑξῆς <ῑ> ἐπὶ τριγώνων ἐστὶν τῶι εἴδει δεδομένων ἄνευ θέσεως. τὰ δὲ ἑξῆς τούτοις ξ̄ ἐπὶ τυχόντων ἐστὶν εὐθυγράμμων χωρίων εἴδει δεδομένων ἄνευ θέσεως. τὰ δὲ ἑξῆς τούτοις ς̄ ἐν

‖ 3 προτεθὲν Greg ‖ 4 ἀληθῶν Ha ἀληθῶς A ‖ 7 προτεθὲν Greg ‖ 10 διορισμὸς — πρόβλημα secl Hu ‖ 11 καὶ om Greg ‖ 15 (ἀποτομῆς) β̄ om Greg ‖ 16 Διωρισμένης Τομῆς add Ha ‖ 19 ante πρὸς add τῶν Hu | ἐπιφανείαι Hu ἐπιφάνειαν A ‖ 20 λβ̄ A λγ̄ Ha λᾱ Greg ‖ 24 καταλέλοιπα Greg κατὰ δὲ λοιπὰ A ‖ 27 πρῶτα Ha πρῶτον A ‖ 28 διαγράμματα secl Hu | τὸ (κʹ) del Hu ‖ 31 ῑ add Greg | τριγώνων Ha τριγώνου A applications of area given in shape. Of the ensuing five, the first is on figures erected upon lines, the other four on triangular areas, that the differences of the squares of the sides are in given ratio to those triangular areas. The next seven, up to the seventy-third, are on two parallelograms, that by the stipulations concerning their angles they are in given ratios to one another. Some of these have similar postscripts on two triangles. Among the next six diagrams, up to the seventy-ninth, two are on triangles, four on more (than two) lines in proportion. The next three are on two lines [that are in ratio that is, and] enclose a given area. The final eight up to the ninetieth are proved on circles, some given only in magnitude, others also in position, that when lines are drawn through a given point, the results are given.

(5) (The Cutting off of a Ratio.)

The proposition of the two books of the *Cutting off of a Ratio* is a single one, albeit subdivided; and therefore I can write one proposition, as follows: through a given point to draw a straight line cutting off from two lines given in position (abscissas extending) to points given upon them, that have a ratio equal to a given one. In fact the figures are varied and numerous, when the subdivision is made, because of the dispositions with respect to each other of the given lines and the various cases of the way that the given point falls, and because of the analyses and syntheses of them and their diorisms. **(6)** Thus the first book of the *Cutting off of a Ratio* contains seven dispositions, twenty-four cases, and five diorisms, three of which are maxima, two minima. There is a maximum in the third case of the fifth disposition, a minimum in the second of the sixth disposition, in the same (number) of the seventh disposition; those in the fourth of the sixth and seventh dispositions are maxima. The second book of the *Cutting off of a Ratio* contains fourteen dispositions, sixty-three cases, and for diorisms those of the first, because it reduces entirely to the first.

If k is a given ratio, (AB) and (CD) are two given coplanar lines, H is a point of the plane formed by (AB) and (CD), which does not belong neither to (AB), nor to (CD), E is a given point of (AB), G a given point of (CD), then determine a line (HKL) secant to (AB) in L and (CD) in K, such as $\dfrac{EL}{GK} = k$.



(64) *(Prop. 21)* Problem for the second (book) of the *Cutting off of a Ratio*, useful for the summation of the fourteenth disposition.

Given two straight lines **AB**, **BΓ**, and producing line **AΔ**, to find a point **Δ** that makes the ratio **BΔ** to **ΔA** the same as that of **ΓΔ** to the excess by which **ABΓ** together exceeds the line that is equal in square to four times the rectangle contained by **AB**, **BΓ**.

The combination cannot be made in any other way, unless **ΔE**, **AΓ** together are equal to the excess **EA**, and all **ΔA** to all **AB**, and furthermore (it is not possible otherwise?) that **EA**, **AΓ**, **ΓB** have the ratio to one another of a square number to a square number, and that **ΓB** is twice **ΔE**.

Let it be accomplished, and let the excess be **AE**;[1] for we have found this in the foregoing (lemma 7.62). Then as is **BΔ** to **ΔA**, so is **ΓΔ** to **AE**.[2] And *alternando*[3] and *separando*[4] and area to area, it follows that the rectangle contained by **BΓ**, **EA** equals the rectangle contained by **ΓΔ**, **ΔE**.[5] But the rectangle contained by **BΓ**, **EA** is given;[6] hence the rectangle contained by **ΓΔ**, **ΔE** too is given.[7] And it lies along **ΓE**, given,[8] exceeding by a square. Hence **Δ** is given (Data 59).[9]

The synthesis will be made thus. Let the excess be **EA**, and along **ΓE** let there be applied the rectangle contained by **ΓΔ**, **ΔE**, exceeding by a square, and equal to the rectangle contained by **BΓ**, **EA**. I say that **Δ** is the point sought. For since the rectangle contained by **BΓ**, **EA** equals the rectangle contained by **ΓΔ**, **ΔE**,[10] therefore putting in ratio[11] and *componendo*[12] and *alternando* as is **BΔ** to **ΔA**, so is **ΓΔ** to **EA**,[13] which is the excess. The same also if we try to take a point making, as **BΔ** to **ΔA**, so **ΓΔ** to the line comprising **ABΓ** together and the line equal in square to four times the rectangle contained by **AB**, **BΓ**. Q.E.D.

Analysis [1]-$AE = AB + B\Gamma - x$, $x^2 = 4(AB \cdot B\Gamma)$

[2]-$B\Delta : \Delta A = \Gamma\Delta : AE$

[3]-Thus $B\Delta : \Delta\Gamma = \Delta A : AE$

[4]-Thus $B\Gamma : \Gamma\Delta = \Delta E : AE$

[5]-Thus $B\Gamma \cdot EA = \Gamma\Delta \cdot \Delta E$

[6]-$B\Gamma \cdot EA$ given

[7]-Thus $\Gamma\Delta \cdot \Delta E$ given

[8]-ΓE given

[9]-Thus Δ given

Synthesis [10]-$\Gamma\Delta \cdot \Delta E = B\Gamma \cdot EA$

[11]-Thus $B\Gamma : \Gamma\Delta = \Delta E : EA$

[12]-Thus $B\Delta : \Delta\Gamma = \Delta A : AE$

[13]-Thus $B\Delta : \Delta A = \Gamma\Delta : EA$

Nella premessa, Pappo osserva come il problema ammetta certamente soluzione con le seguenti lunghezze di segmenti:

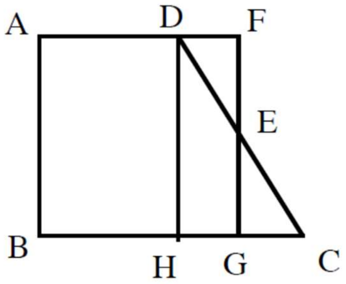
Δ		E		A		Γ		B
	8		9		1		16	

Dalla *Metrica* di Erone d'Alessandria: un esempio di analisi e sintesi

Problema 10, Libro I

Sia ABCD un trapezio rettangolo avente gli angoli in A, B retti e sia il lato AD di 6 unità, il lato BC di 11 e il lato AB di 12 unità. Trovare la sua area ed anche CD.

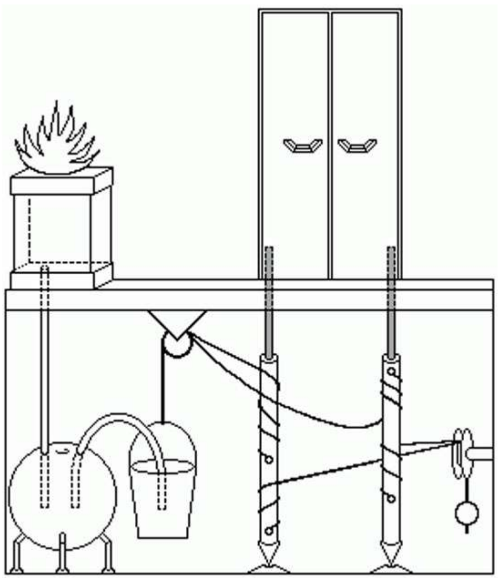
Si divida CD in due parti uguali in E e da E si conduca la retta FEG, parallela ad AB, e il lato AD sia prolungato fino a F.



Poiché DE è uguale ad EC, DF è uguale a GC. A questi (due ultimi segmenti) si aggiungono, come prolungamenti, AD; ne consegue che AF, BG, presi insieme, sono uguali ad AD, BC presi insieme. Ora, AD, BC, presi insieme, sono **dati**, poiché lo sono singolarmente. Dunque AF, BG, presi insieme, sono anch'essi **dati**, ossia sono **dati** due BG, e pertanto è **dato** BG.
Ma è **dato** anche AB; quindi il parallelogramma ABFG è **dato**. E poiché il triangolo DEF è uguale ad EGC, una volta aggiunto il poligono a cinque lati ABGED, si ottiene il parallelogramma ABFG, uguale al trapezio ABCD. Ma è stato dimostrato che il parallelogramma ABFG è **dato**; quindi anche il trapezio ABCD è **dato**.

E CD si troverà così: si conduca la perpendicolare DH. Poiché AD è **dato**, anche BH è **dato**; ma anche BC; e dunque anche il rimanente CH è **dato**. Ma lo è anche DH — in quanto uguale ad AB – e l'angolo in H è retto; quindi anche CD è **dato**.

E ciò **sarà sintetizzato seguendo l'analisi** nella maniera seguente:
Aggiungi 6 a 11: ottieni 17. Prendine la metà: ottieni 8 2'. Moltiplica per 12: ottieni 102. Questa è l'area.
E per CD, procedi così: sottrai 6 da 11: ottieni 5. Moltiplicalo per se stesso: ottieni 25. Moltiplica 12 per se stesso: ottieni 144. Aggiungi 25: ottieni 169. Il suo lato è 13. Questo sarà CD.



Erone e la *Pneumatica* (apparecchio per aprire le porte del tempio)

The Analytic Art

Nine Studies in Algebra, Geometry and Trigonometry from the *Opus Restitutae Mathematicae Analyseos, seu Algebrâ Novâ*

François Viète

Translated by
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INTRODUCTION TO THE ANALYTIC ART[1]

CHAPTER I

On the Meaning and Components of Analysis and on Matters Useful to Zetetics

There is a certain way of searching for the truth in mathematics that Plato is said first to have discovered. Theon called it analysis, which he defined as assuming that which is sought as if it were admitted [and working] through the consequences [of that assumption] to what is admittedly true, as opposed to synthesis, which is assuming what is [already] admitted [and working] through the consequences [of that assumption] to arrive at and to understand that which is sought.[2]

Although the ancients propounded only [two kinds of] analysis, zetetics and poristics,[3] to which the definition of Theon best applies, I have

[1]The title varies slightly in the different editions of this work: 1591, 1624, and 1631 have *In Artem Analyticem Isagoge,* 1635 has *In Artem Analyticam Isagoge,* and 1646 has *In Artem Analyticen Isagoge.*

[2]T. L. Heath, in vol. III, p. 442, of his second edition of Euclid's *Elements* (Cambridge, The University Press, 1925) points out that these definitions were interpolated in Book XIII before Theon's time and have been variously attributed to Theaetetus, Eudoxus, and Heron. See also the definitions of the same terms by Pappus as translated by Heath in his essay on "Mathematics and Astronomy" in *The Legacy of Greece,* ed. R. W. Livingstone (Oxford University Press, 1921), p. 102.

[3]*ζητητιὴν και ποριστικὴν.* Viète apparently borrowed these two terms, but not the meanings he attributes to them, from Pappus who, in *The Treasury of Analysis,* said: "Now analysis is of two kinds, one, whose object is to seek the truth [ζητητικὸν], being called theoretical, and the other, whose object is to find something set for finding [ποριστικὸν], being called problematical. . . ." Quoted in *Selections Illustrating the History of Greek Mathematics,* tr. Ivor Thomas (Cambridge, Mass., Loeb Classical Library, 1941), p. 599. Compare Beaugrand's notes to his edition of this work of Viète's (1631, p. 25): "Porro Analysis veterum duplex, una theorematica, qua Theorematis oblati veritas examinatur. Altera Problematica, cuius dua sunt partes; prior qua propositi Problematis solutio inquiritur Zetetice vocatur;

added a third, which may be called rhetics or exegetics.[4] It is properly zetetics by which one sets up an equation or proportion[5] between a term that is to be found and the given terms, poristics by which the truth of a stated theorem is tested by means of an equation or proportion,[6] and exegetics by which the value of the unknown term in a given equation or proportion is determined. Therefore the whole analytic art, assuming this three-fold function for itself, may be called the science of correct discovery in mathematics.

Now whatever pertains to zetetics begins, in accordance with the art of logic, with syllogisms and enthymemes the premises of which are those

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posterior quae determinat quando, qua ratione, et quot modis fieri possit Problema Poristice dici potest." This definition is picked up and followed by Durret in the notes to his translation (p. 6), by Jacques Ozanam in his *Dictionaire Mathematique* (Amsterdam, 1691) and, as far as it concerns poristics, by Alexandre Saverein in his *Dictionnaire Universel de Mathematique et de Physique* (Paris, 1753), vol. II, p. 314.

[4]The vagaries of sixteenth-century punctuation and the ambiguity of the word *constitui* make the reading of the end of this sentence and the beginning of the next uncertain. In the Latin we have *constitui tamen etiam tertiam speciem, quae dicitur* ῥητικὴ ἢ ἐξηγητικὴ *consentaneum est, ut sit Zetetice qua invenitur,* etc. An alternative reading to the one adopted above, would be, ". . . it is proper to add a third type which may be called rhetics or exegetics. Hence it is zetetics by which. . . ." Ritter, Vasset, and Smith so read the passage; Vaulezard and Durret read it as given above.

[5]1624 has *aequalitas proportione,* an error for *aequalitas proportiove.*

[6]*Poristice, qua de aequalitate vel proportione ordinati Theorematis veritas examinatur.* The question arises whether Viète is speaking of testing a theorem derived from an equation or proportion or of testing a theorem by means of an equation or proportion. Either fits his language and its context. Vaulezard translates this passage, "Le Poristique, par lequel est enquis de la verité du Theoreme ordonné, par l'egalité ou proportiõ"; Vasset, "La Poristicque est celle par laquelle on examine la verité d'un Theoreme déja ordonné, par le moyen de l'égalité ou proportion"; Durret, "La Poristique, celle par le moyẽ de laquelle on examine la verité du Theoréme ordonné touchant l'egalité, ou proportion"; Ritter, "par la méthode Poristique on examine, au moyen de l'égalité ou de la proportion, la verité d'un théorème enoncé"; and Smith, "a poristic art by which from the equation or proportion the truth of the theorem set up is investigated." Vaulezard offers a further explanation that the task of poristics is to "examiner & tenter si les Theoremes & consequences trouvées par le Zetetique sont veritables." Compare the passage from Beaugrand, n. 3 supra, and the illustrations he gives on pp. 75ff. of his edition of Viète's work.

Thomas Harriot, in his *Artis Analyticae Praxis* (London, 1631), p. 2, throws a little further light on his century's understanding of the difference between the zetetic and the poristic processes: "Veteres Analystae praeter Zeteticen quae ad problematum solutionem proprie pertinet aliam Aanlycices [sic] speciem fecerunt poristicen. . . . Methodus enim utriusque Analytica est, ab assumpto probando tanquam concesso per consequentia ad verum concessum. In hoc tamen inter se differunt, quod Zetetice quaestionem deducit ad aequale datum scil. quaesito, poristice autem ad idem, vel concessum. . . . Unde et altera inter eas oritur differentia quod in poristice, cum processus eius terminetur in identitate vel concesso, ulterior resolutione non sit opus (ut fit in Zetetice) ad propositi finalem verificationem."

No work of Viète's on poristics is extant and there is no certainty that he ever wrote one.

fundamental rules[7] with which equations and proportions are established. These are derived from axioms and from theorems created by analysis itself. Zetetics, however, has its own method of proceeding. It no longer limits its reasoning to numbers, a shortcoming of the old analysts, but works with a newly discovered symbolic logistic[8] which is far more fruitful and powerful than numerical logistic for comparing magnitudes with one another. It rests on the law of homogeneous terms first and then sets up, as it were, a formal series or scale of terms ascending or descending proportionally from class to class in keeping with their nature[9] and, [by this

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At two places in his work on *A Supplement to Geometry*, however, (p. 388ff. infra) he uses the expression *inventum est in Poristicis* with the possible implication that there was once such a work. It is not out of the question that he treated of poristics at length in the now-lost *Ad Logisticem Speciosam Notae Posteriores*.

[7] *symbola*

[8] *per logisticem sub specie*. In Chapter III this becomes *Logistice speciosa* (algebra) in contrast to *Logistice numerosa* (arithmetic). On the history of the word "logistic," see David Eugene Smith, *History of Mathematics* (Boston, 1925), vol. II, pp. 7, 392, and Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, tr. Eva Brann (Cambridge, Mass., 1968), passim.

Viète's curious words *sub specie* and *speciosa* have called forth a variety of comments and explanations: One, by John Wallis in his *Treatise of Algebra* (London, 1685), p. 66, is to the effect that Viète's use of *species* reflects his familiarity with the civil law where the word, Wallis says, is used to designate unknown or indefinite defendants in what we today would call "John Doe" cases; Wallis's view appears to be an expansion of that of Harriot, op. cit., supra n. 6, p. 1, that the meaning of the phrase *in specie* derives *ex usu forensi recepto speciei vocabulo*. Another, by Samuel Jeake in his Λογιστικηλογία (London, 1696), p. 334, has it that this "name . . . with the Latins serveth for the Figure, Form or shape of any thing" and that, accordingly, "*Species* are Quantities or Magnitudes, denoted by Letters, signifying Numbers, Lines, Lineats, Figures Geometrical, *&c.*" Alexandre Saverein's *Dictionnaire Universel de Mathematique et de Physique* (Paris, 1753), vol. I, p. 17, says that the expression "algébre spécieuse" derives from that fact that quantities are represented by letters which designate "leur forme et leur espece," adding "d'ou vient le mot spécieuse." Ritter (p. 232, n. 3), on the other hand, thinks Viète coined a new meaning for an old word, the new meaning having no connection with its meanings in Latin or French. Still another explanation is offered in such modern French dictionaries as Littré's, for example, where the word "spécieux" is said to come directly from the Latin *speciosa* with its meaning of "beautiful in appearance" and the phrase "Arithmétique spécieuse" is explained by saying that it is "ainsi dite à cause de la beauté de l'algèbre par rapport à l'arithmétique." Smith thinks Diophantus "the most likely source for Vieta's use of the word 'species' " and that it is, in effect, his substitute for Diophantus' εἶδος. I am inclined to believe that Viète chose to give the noun *species*, with its meanings of "appearance," "semblance," "likeness," etc. and no doubt with an appreciation of its ancillary overtones, the somewhat enlarged meaning of a representation or symbol and have translated accordingly.

[9] *ex genere ad genus vi sua proportionaliter*. The phrase *vi sua proportionaliter* in this context is troublesome. Vaulezard translates it as "de leur propre puissance," Vasset as "d'elles-meme proportionellement," Durret as "proportionellement par leur force," Ritter as "proportionellement pour leur propre puissance," and Smith as "by their own nature."

series,] designates and distinguishes the grades and natures of terms used in comparisons.

CHAPTER II

On the Fundamental Rules of Equations and Proportions

Analysis accepts as proven the well-known fundamental rules of equations and proportions that are given in the *Elements*. They are these:

1. The whole is equal to [the sum of] its parts. Nozione Comune 8
2. Things equal to the same thing are equal to each other. Nozione Comune 1
3. If equals are added to equals, the sums are equal. Nozione Comune 2
4. If equals are subtracted from equals, the remainders are equal. Nozione Comune 3
5. If equals are multiplied by equals, the products are equal.
6. If equals are divided by equals, the quotients are equal.
7. Whatever are in proportion directly are in proportion inversely and alternately.
8. If similar proportionals are added to similar proportionals, the sums are proportional.
9. If similar proportionals are subtracted from similar proportionals, the remainders are proportional.
10. If proportionals are multiplied proportionally, the products are proportional.
11. If proportionals are divided proportionally, the quotients are proportional.
12. An equation or ratio is not changed by common multiplication or division [of its terms].
13. The [sum of the] products of the several parts [of a whole] is equal to the product of the whole.
14. Consecutive multiplications of terms and consecutive divisions of terms yield the same results regardless of the order in which the multiplication or division of the terms is carried out.[10]

A sovereign rule,[11] moreover, in equations and proportions, one that is of great importance throughout analysis, is this:

[10] *Facta continue sub magnitudinibus, vel ex iis continue orta, esse aequalia quocumque magnitudinum ordine ductio vel adplicatio fiat.*

[11] *κύριον . . . symbolum.*

15. If there are three or four terms such that the product of the extremes is equal to the square of the mean or the product of the means, they are proportionals. Conversely,

16. If there are three or four terms and the first is to the second as the second or third is to the last, the product of the extremes will be equal to the product of the means.

Thus a proportion may be said to be that from which an equation is composed and an equation that into which a proportion resolves itself.[12]

CHAPTER III

On the Law of Homogeneous Terms and on the Grades and the Kinds of Magnitudes of Comparison

[1] The prime and perpetual law of equations or proportions which, since it deals with their homogeneity, is called the law of homogeneous terms, is this:

Homogeneous terms must be compared with homogeneous terms,[13]

for, as Adrastos said,[14] it is impossible to understand how heterogeneous terms [can] affect each other. Thus,

If one magnitude is added to another, the latter is homogeneous with the former.

If one magnitude is subtracted from another, the latter is homogeneous with the former.

If one magnitude is multiplied by another, the product is heterogeneous to [both] the former and the latter.

[12] *Itaque proportio dici cõstitution aequalitatis. Aequalitas, resolutio proportionis.* This cryptic sentence summarizes a good deal of Viète's approach to algebra, as will become apparent later on. In addition, the word *constitutio* is one of his favorites. Vasset and Ritter translate it in this place by "établissement" or "establissement," Vaulezard and Durret by "constitution," and Smith by "composition." In many other places in this book, I have rendered it by "structure" or the like.

[13] *Homogenea homogeneis comparari.*

[14] Viète's source for Adrastos's dictum was probably Theon's Euclid. It is quoted by Jacob Klein, op. cit. supra n. 8, p. 276, n. 253. See p. 173 for Klein's appraisal of the use Viète makes of it. On Adrastos himself—he lived in Aphrodias in the first half of the second century—see George Sarton, *Introduction to the History of Science* (Baltimore, 1927), vol. I, p. 271.

If one magnitude is divided[15] by another, [the quotient] is heterogeneous to the former [i.e., to the dividend].

Much of the fogginess and obscurity of the old analysts is due to their not having been attentive to these [rules].

2. Magnitudes that ascend or descend proportionally in keeping with their nature from one kind to another are called scalar terms.

3. The first of the scalar magnitudes is the side or root.[16] [Then follow:]

2. The square
3. The cube
4. The square-square
5. The square-cube
6. The cubo-cube
7. The square-square-cube
8. The square-cubo-cube
9. The cubo-cubo-cube

and so on, naming the others in [accordance with] this same series and by this same method.[17]

4.[18] The kinds of magnitudes of comparison,[19] naming them in the same order as the scalar terms, are:

1. Length or breadth
2. Plane
3. Solid
4. Plano-plane

[15]*adplicatur*, Viète's usual term for the verb "divide," though he sometimes uses *dividere*. Durret (p. 14) comments on the difference between "application" and "division" thus: "car l'application differe de la division, en ce que le genre de la grandeur engendrée, ou quotient, est tousiours heterogene au genre de la grandeur appliquée; mais au contraire le quotient de la division est tousiours homogene au genre de la grandeur divisée." In the latter case, for instance, the division of a line into, say, three parts, gives three lines that are homogeneous with the original line, whereas the "application" of a plane by a length gives another length which is not homogeneous with the plane.

[16]*Latus, seu Radix*. Viète's more usual term is *latus*. Elsewhere he uses *radix* with a somewhat different meaning; see n. 54 infra.

[17]In most places in this translation, I have replaced Viète's nomenclature by the more familiar terms "first power" . . . "fourth power," "fifth power," etc., or, when his terms are attached to letters, by the use of numerical exponents in the modern form.

[18]In the text this and the next three paragraphs are misnumbered 7, 8, 9 and 10.

[19]*magnitudinum comparatorum*. Viète usually uses *homogeneum comparationis* for the singular form of this expression. In either case it means the purely numerical terms with which the variable terms are equated or compared. The same length-plane-solid-etc. terminology that Viète uses here is also used by him for his coefficients, but he calls these *subgraduales*.

5. Plano-solid
6. Solido-solid
7. Plano-plano-solid
8. Plano-solido-solid
9. Solido-solido-solid

and so on, naming the others in [accordance with] the same series and by the same method.[20]

5. In a series of scalar terms, the highest, counting up from the root, is called the power. The term of comparison [must be] consistent with this. The other lower scalar terms are [referred to as] lower-order terms.[21]

6. A power is pure when it lacks any affection. It is affected when[22] it is associated [by addition or subtraction] with a homogeneous term that is the product of a lower-order term and a supplemental term [or] coefficient.[23]

7. A supplemental term the product of which and a lower-order term is homogeneous with the power it [i.e., the product] affects is called a coefficient.[24]

CHAPTER IIII[25]

On the Rules of Symbolic Logistic

Numerical logistic is [a logistic] that employs numbers, symbolic logistic one that employs symbols or signs for things[26] as, say, the letters of the alphabet.

[20]Later on it will often be convenient to abbreviate these rather clumsy terms by showing them as exponents. For instance *B plano-solidum* will appear as B^{ps} and *X solido-solidum* as X^{ss}, and so forth.

[21]*gradus parodici ad potestatem.*

[22]The text has *cui*, which I read as a misprint for *cum*.

[23]*adscita coefficiente magnitudine.* Ritter translates this as "une grandeur étrangère coefficiente," Vasset as "une grandeur coëfficiente empruntée," Vaulezard as "une grandeur adscitice coeficiente," and Durret as "la grandeur coeficiente adiointe."

[24]*Subgraduales.* I take it that, rather than using *sub* to indicate that the "subgradual" is of lower degree than the "gradual," Viète here uses it to indicate multiplication (cf. n. 29 infra)—that is, a "subgradual" is a multiplier of a "gradual," i.e., of a degree of the unknown lower than the power.

[25]1591 and other early editions of Viète's works use this form of the Roman numeral.

[26]*Logistice numerosa est quae per numeros, Speciosa quae per species seu formas exhibitur.* The translations of this passage vary greatly. Vasset has "La Logistique nombreuse est celle qui s'exerce par les nombres. Et la specieuse est celle qui se pratique par les especes ou

There are four basic rules for symbolic logistic just as there are for numerical logistic:

Rule I

To add one magnitude to another

Let there be two magnitudes, *A* and *B*. One is to be added to the other.

Since one magnitude is to be added to another, and homogeneous and heterogeneous terms do not affect each other, the two magnitudes proposed are homogeneous. (Greater or less do not constitute differences in kind.) Therefore they will be properly added by the signs of conjunction or addition and their sum will be *A* plus *B*, if they are simple lengths or breadths. But if they are higher up in the series set out above or if, by their nature, they correspond to higher terms, they should be properly designated as, say, A^2 plus B^p, or A^3 plus B^s, and so forth for the rest.

Analysts customarily indicate a positive affection by the symbol +.

Rule II

To subtract one magnitude from another

Let there be two magnitudes, *A* and *B*, the former the greater, the latter the less. The smaller is to be subtracted from the greater.

Since one magnitude is to be substracted from another and homogeneous and heterogeneous magnitudes do not affect one another, the two given magnitudes are homogeneous. (Greater or less do not constitute differences in kind.) Therefore the subtraction of the smaller from the larger is properly made by the sign of disjunction or subtraction, and the disjoint terms will be *A* minus *B* if they are only simple lengths or breadths. But if they are higher up in the series set out above or if, by their nature, they correspond to higher terms, they should be properly designated as, say, A^2 minus B^p, or A^3 minus B^s, and so forth for the rest.

The process is no different if the subtrahend is affected, since the whole and its parts ought not to be thought of as being subject to different

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formes, mesmes des choses"; Vaulezard has "Le Logistique Numerique est celui qui est exhibé & traité par les nombres, le Specifique par especes ou formes des choses"; Durret has "La logistique nombreuse est celle, qui se fait par les nombres; la specieuse, par les especes, ou formes des choses"; Ritter has "Logistique numérale est celle qui est exposée par des nombres. Logistique spécieuse est celle qui est exposée par des signes ou de figures"; and Smith has "The numerical reckoning operates with numbers; the reckoning by species operates with species or forms of things."

The Practice of the Analytic Art

For solving Algebraic equations by a new, convenient and general method:

A Treatise

Transcribed with the utmost accuracy and care from the last papers of Thomas Harriot, the celebrated Philosopher and Mathematician:

and Dedicated to the Most Illustrious Lord
Lord Henry Percy
Earl of Northumberland

Who ordered this work to be newly revised, transcribed, and published for the general use of Mathematicians—a work which was first composed for his own use, under the auspices of his Generosity and Patronage, and therefore a dedication which is most richly deserved.

London

by Robert Barker, Royal Printer:
And Successor to John Bill, in the Year 1631.

Definitions

Certain definitions, which (in lieu of an introduction) may assist in understanding the terms in common use in the art, as well as those peculiar to the present treatise **1**

DEFINITION 1 [1]

Specious Logistic: This type of Arithmetic is frequently used and is absolutely necessary in these writings on Analysis; it is a sibling of Arithmetic, through participation in the same genus. For Arithmetic is numerical Logistic. The distinction between them goes no further than that signified by their names: in Arithmetic, the quantities of measurable things are expressed and reckoned by characters or figures peculiar to the art, by numerals, as in measurement generally; in the former, however, the quantities themselves are indicated and in every way handled through written signs—the letters of the alphabet, that is—'speciously', as it were (borrowing the term 'specious' from commercial usage). Hence it has received the name Specious.

DEFINITION 2

Equation is used in its common sense for any sort of equality of two or more quantities; but as a special term of this art, it is the clearly determined equality of the sought quantity with some given quantity, when a comparison has been made of one with the other. The part which is sought is a simple or affected (conditioned) power but the part given is commonly called the given homogeneous term of the comparison or equation.

DEFINITION 3 [2]

In propositions of any sort and in drawing up theorems or problems from them scientifically, the best method of proof and an entirely natural way, is that by

Bold numbers in the margin refer to the pages of the 1631 printed edition of the *Praxis*. The preface was unpaginated.

which one proceeds from the first principles and elements proper to each subject, composing a chain of logical steps until the proposition is confirmed. Hence it is called the compositive method or, in the terminology of the ancient practitioners, *Synthesis*.

Definition 4 [3]

Yet, in solving problems—especially those encountered by chance—the Arithmetician very often has no recourse to suitable middle stages for proving the proposition; thus, if he takes the natural route of synthesis from first principles and elements of the science, he cannot make any progress in finding and demonstrating a solution by logical means. And so, in such a case of ignorance (which happens almost constantly) necessity teaches him to resort to a reverse procedure, the opposite of the natural one. He makes a start from some unknown and sought quantity which occurs in the problem, as if it were a known and given quantity; he then proceeds by resolving it in a chain of logical steps, until he reaches an equality between the quantity which was taken as given (standing alone, or raised to a power and qualified by coefficients) and some quantity which really is given. If this equality can be found and properly established by such a technique, then, from this equality, the sought quantity itself either will be self-evident, or can be elicited through a further application of the technique; and so the problem can, at length, be solved. And that is the method of resolution which the ancients called by the descriptive term *Analysis*.

Definition 5

2 The words 'composition' and 'resolution' which were introduced in these two definitions are the usual ones Mathematicians employ, by which they signify explicitly, when necessary, the two contrary paths of reasoning used in constructing demonstrations. In the first, one descends from the more simple and less composite to the more composite—that is, by composition—following the natural structure and ordering of the sciences: in other words, from the prior to the posterior. In the second, however, one ascends to the conclusion from the more composite to the less composite and the simple—that is, by resolution—in a reverse order, contrary to the natural one: in other words, from the posterior to the prior. For, the elements and axioms of the sciences, if they are legitimately constituted, ought to be convertible; hence it comes about that those which in nature are antecedents can, in reasoning, be consequents. It is for this reason that logical progress through deductive steps is, of necessity, equally firm and demonstrative in both directions.

Definition 6 [4]

From the above definition of Analysis it may be inferred that it consists of two parts, distinguished by function. The first part is wholly concerned with establishing equalities. As has been said, it begins from the thing which is sought, assuming it as if it were given; then it aims at finding and establishing, by logical consequences, an equality between the sought quantity (which was assumed as known) and some given quantity; and, having established the equality, it terminates and rests there. And since this sort of establishment of equalities consists in the skilled examination of the sought quantity, the ancients called this art *Zetetic*, meaning an 'investigative' or 'inquisitive' art.

Definition 7 [5]

The second part of Analysis has this purpose: from an equality established by Zetetic, the sought quantity is revealed by a continued or a modified type of resolution: either by a symbol (if it can be revealed by an algebraic expression), or by a numeral (if a numerical solution is required)—and so, at length, a complete solution to the proposed problem is established. François Viète, the great master of the art of Analysis, called this part of Analysis *Exegetic*, meaning 'declaratory' or 'evidential'.

Definition 8 [6]

The ancient Analysts, in addition to Zetetic (which is specifically concerned with the solution of problems) distinguished another kind of Analysis: *Poristic*, meaning 'deductive'. This is used to examine whether theorems put forward by chance are true or not. The method of both (zetetic and poristic) is Analytic, the proof proceeding by logical steps from something assumed as given to some given truth. They differ, however, in this: Zetetic leads the investigation to an equality (an equality, that is, of a given quantity with the sought quantity), whereas Poristic brings it to an identity, or to a given quantity itself, as one can see in examples of each kind. Hence there arises a further difference between them: in Poristic, since its progress is brought to an end by identity or by a given quantity, there is no need of a further process of resolution (as there is in Zetetic) for the final verification of the proposition.

Definition 9

From the definition of Exegesis already given, it is evident that it needs to be twofold, in accordance with the twofold nature of Logistic—numerical and specious. Although these deal with the same subject-matter, and have the same end in view—the resolution of equalities which have been already established—in practice, however, and in mode of operation, they fall into entirely different genera. For, specious Exegesis takes the equality first established by means of Zetetic; then, by a continuous chain of inferences, it reduces it to a symbol or formal expression which is suitably arranged for resolution; and finally, from this properly ordered expression, by means of a precise and straightforward technique, it reveals the sought quantity in its essential form. Indeed, when this form of Exegesis is applied in equations which do not go beyond the quadratic order but remain within a plane locus (as the ancients say), it should be regarded as perfectly scientific, because of the precise uniformity and determined nature of the method. When certain mo- **3**
dern writers on Algebra have tried, however, to advance the art to the solution of higher order equations (that is, cubic and biquadratic) which are beyond the range of the previous method, they have left the method to us in a mutilated and

incomplete state, because of ineradicable faults in their fundamental principles; as a consequence, a large proportion of the equations which fall within these higher categories have been regarded as insoluble.

DEFINITION 10

And so that other Exegesis—*numerical Exegesis*—was devised, which extends to the resolutions of all equations of all orders. It employs a general and infallible method: proceeding from any equation whatsoever which has been established by the resolving process of Zetetic and then returned to numerical form, the sought quantity is, by this method, revealed numerically by a secondary application of a different sort of resolution. This Exegesis has an art all of its own, provided with its own rules and instructions for practical application; they are given in the present treatise, which is wholly concerned with Exegesis.

DEFINITION 11

It seemed appropriate to call the numerical resolution of Exegesis *secondary*; secondary, that is, to the antecedent Zetetic resolution, which one secures first. In this way, by adding a qualification to the name that they share, it will be clear that both are a form of analysis or resolution, even if they are of different genera: zetetic is logical or discursive [resolution], while Exegesis, on the other hand, is instrumental. For, numerical Exegesis is precisely this: an elevation (so to speak) of the ancient practice of Arithmetic (which, until now, has gone no further in its usual application than the extraction of roots of only the simplest powers) to a new generality of method, not even attempted by the ancients.

DEFINITION 12

The word *root* is used in two senses in what follows. When establishing equalities through Zetetic, that which was assumed or taken as given at the beginning of the reasoning is, in fact, constantly sought; thus, in equations that have already been established, it can be called the sought or posited root—for as long, that is, as it remains concealed beneath the veils of the equation either in the form of a power or simply as itself. In resolving equations, on the other hand, or dealing with equations which have been already resolved—when it has been revealed by specious exegesis from an equation already established, by analytic reasoning and in its algebraic form; or when it has been elicited by numerical exegesis from the given homogeneous term of the equation, by analytic operation and in explicit numerical form—one may call it the revealed or extracted root (the variation in name corresponding to the variation in the type of resolution); this has also commonly come to be called the *value* of the sought root.

And so although in Vite's Exegesis, where problems about quadratics and cubes and higher powers are clearly dealt with, one inevitably had to inquire about the root of the power, in this work—which deals not with the powers, but with the equations themselves taken subjectively and in an integral form, in which the root is no more [the root] of the highest power than it is of the lower degrees—it seemed more in keeping with the stated intentions if the inquiry was directed to the explicatory root of the equation, or at least to the value of the sought root. Since, however, the explicatory root of the equation also generates the power, it is obvious that, to those who understand the matter itself, it will make no difference at all whether it is given this name or that.

Definition 13

The definition given above of an equation should be taken as applying to one that has properly been established by Zetetic. Since such an equation is commonly 4
derived from the terms of problems, or of questions put forward in whatever other way, it may be called a *common* or *adventitious* equation; this is so that its name may distinguish it from the other, entirely different kind of equation, which will be called 'canonical'—and about which more will be said presently.

Definition 14

There is also another kind of equations: those which, although not canonical, will be called *originals of canonical equations* in what follows, because canonical equations are derived from them, as from their originals. Equations of this kind are formed immediately, and without any discursive reasoning, from binomial roots by genesis or multiplication. In these cases, whatever is formed by multiplying the roots, is then set equal to those same roots (as they were before they were multiplied and only arranged in the proper way for multiplication), as one can see in these examples here.

$$\left.\begin{array}{c} a+b \\ a-c \end{array}\right| = \begin{array}{l} aa+ba \\ \quad -ca-bc \end{array}$$

$$\left.\begin{array}{c} a+b \\ a+c \\ a-d \end{array}\right| = \begin{array}{l} aaa+baa+bca \\ \quad +caa-bda \\ \quad -daa-cda-bcd \end{array}$$

$$\left.\begin{array}{c} a+b \\ a+c \\ a+d \\ a-f \end{array}\right| = \begin{array}{l} aaaa+baaa+bcaa \\ \quad +caaa+bdaa \\ \quad +daaa+cdaa+bcda \\ \quad -faaa-bfaa-bcfa \\ \qquad -cfaa-bdfa \\ \qquad -dfaa-cdfa-bcdf \end{array}$$

The form of these equations does not agree at all with the definition of a proper equation. But the two sorts of canonical equation which are derived from these, namely the primary and secondary, both conform to the definition, and are properly applicable in practice.

Definition 15

The *primary* sort of *canonical equations* is of those which are established by derivation from original equations. The formal arrangement of the roots (the first part of an original equation) is suppressed; then, from the homogeneous terms of the second part, a given homogeneous term is transferred to the side opposite to the rest, by changing its sign—and thus a primary canonical equation is formed. The mode of derivation is set out in Section 2, but here are examples of the form.

$$\begin{aligned} &aa + ba \\ &\quad - ca = +bc \end{aligned}$$

$$\begin{aligned} &aaa + baa + bca \\ &\quad + caa - bda \\ &\quad - daa - cda = +bcd \end{aligned}$$

$$\begin{aligned} &aaaa + baaa + bcaa \\ &\quad + caaa + bdaa \\ &\quad + daaa + cdaa + bcda \\ &\quad - faaa - bfaa - bcfa \\ &\qquad\qquad - cfaa - bdfa \\ &\qquad\qquad - dfaa - cdfa = +bcdf \end{aligned}$$

DEFINITION 16

5 The *secondary* sort of *canonical equations* is of those which are established by reduction from primary canonical equations. By removing any one of the incidental degrees it becomes a secondary or reduced one. This kind of reduction takes various forms, and they are discussed in the third section of this treatise; but some examples of this type follow:

$$aa = +bb$$

$$\begin{aligned} &aaa - bba \\ &\quad - bca \\ &\quad - cca = + bbc \\ &\qquad\qquad\quad + bcc \end{aligned}$$

$$\begin{aligned} &aaaa - bbaa - bbca \\ &\quad - bcaa - bcca \\ &\quad - ccaa - bbda \\ &\quad - bdaa - bdda \\ &\quad - cdaa - ccda \\ &\quad - ddaa - cdda \\ &\qquad\qquad - 2.bcda = + bbcd \\ &\qquad\qquad\qquad\qquad + bccd \\ &\qquad\qquad\qquad\qquad + ccdd \end{aligned}$$

DEFINITION 17

These two sorts of equations are regarded as canonical because, when they are applied as canons or yardsticks, the number of roots in common equations is determined (which can be seen in the Fifth Section); hence the name 'canonical' should be given to them not from the form in which they are established but from their use as instruments in this way.

DEFINITION 18

An equation is called reciprocal when a given homogeneous term is equated with one made from coefficients: and, reciprocally, when the [highest] power is equated with a term made from incidental [i.e., lower] degrees. As, for example, $aaa - caa + bba = +bbc$. For bbc is equated with $\left.\begin{array}{c} bb \\ c \end{array}\right|$ and aaa is equated with $\left.\begin{array}{c} aa \\ a \end{array}\right|$.

Division of this Analytic Treatise 6

In two parts of which

- the first is preparatory to the exegesis, and is divided into 6 sections, namely...
  - the first: the forms of Specious Arithmetic are set out, together with practical examples.
  - the second: the derivation of primary canonical equations from their originals is demonstrated. Prefaced to this is a systematic description of the generation of the originals from binomial roots.
  - the third: this treats the reduction of secondary canonical equations from primary ones, by the removal of an incidental degree, while the substituted root remains unchanged.
  - the fourth: the explicatory roots of canonical equations (both primary and reduced) are indicated.
  - the fifth: the number of roots of common equations is determined by means of equipollent canonicals.
  - the sixth: this treats the reduction of common equations by the exclusion of an incidental degree and by the change of the substituted root.
- the second contains the actual practice of numerical exegesis, explained through rules and examples; inasmuch as this is the principal skill of this art, anything treated in the previous part must be understood to be subordinate to this.

Section One[1] 7

The Forms of the Four Operations of Specious Arithmetic[2] Illustrated by Example[3]

Examples of Addition

To be added	a	To be added	aa	To be added	aaa
	b		bc		bcc
Total	$a + b$	Total	$aa + bc$	Total	$aaa + bcc$

To be added	$a + b$	To be added	$a + b$
	$c + d$		$c - d$
Total	$a + b + c + d$	Total	$a + b + c - d$ [4]
To be added	$a + b$	To be added	$a + b$
	$-d$		$-b$
Total	$a + b - d$	Total	a
To be added	$a + b$	To be added	$aa + cc$
	$c + b$		$aa + cc$
Total	$a + c + 2b$	Total	$2aa + 2cc$ [5]
To be added	$aaa + cdf - ddd$	To be added	$b + 7a$
	$aaa + bdd + ddd$		$+9a$
Total	$2aaa + cdf + bdd$	Total	$b + 16a$

To be added	$b + 7a$	To be added	$b + 9a$	To be added	$b - 9a$
	$-9a$		$b - 7a$		$b + 7a$
Total	$b - 2a$	Total	$2b + 2a$	Total	$2b - 2a$

Examples of Subtraction

Given	a	Given	aa	Given	aaa
To be subtracted	b	To be subtracted	bc	To be subtracted	bcc
The remainder	$a - b$	The remainder	$aa - bc$	The remainder	$aaa - bcc$

Il riferimento seguente è all'opera perduta del matematico ed astronomo greco Apollonio di Perga (III-II sec. a.C.), intitolata *I luoghi piani.* Il suo trattato principale è dedicato alle *Sezioni coniche*.

(21) Two (books) of Plane Loci:

Of the loci in general, some are fixed, as Apollonius also states before his own elements: the locus of a point being a point, a line the locus of a line, a surface of a surface, a solid of a solid; others are path loci: as a line of a point, a surface of a line, a solid of a surface; others are domain loci: as a surface of a point, a solid of a line.

(22) Among the (loci) in the *Domain of Analysis,* those of things given in position are fixed, while the so-called 'plane' and 'solid' and 'curvilinear' (loci) are path loci of points, and the loci on surfaces are domain loci of points, but path loci of lines. However, the curvilinears are demonstrated on the basis of the (loci) on surfaces. The loci about which we are teaching, and generally all that are straight lines or circles, are called 'plane'; all those that are sections of cones, parabolas or ellipses or hyperbolas are called 'solid'; and all those loci are called 'curvilinear' that are neither straight lines nor circles nor any of the aforesaid conic sections. The loci that Eratosthenes named 'with respect to means' are in classification among those named above, but they have been named on the basis of the characteristic of their hypotheses.

(23) The ancients compiled their elements attending to the order of these plane loci; but the people who came after them disregarded this, and added others — as if they were not boundless in number if one wanted to add some that do not belong to that order! Hence I shall put the additional ones later, and those that belong to the order first, encompassing them by one proposition, namely: (1) If two straight lines are drawn either from one given point or from two, and either in a straight line or parallel or containing a given angle, and either holding a ratio to one another or containing a given area, and the end of one touches a plane locus given in position, the end of the other will touch a plane locus given in position, sometimes of the same kind, sometimes of the other, and sometimes similarly situated with respect to the straight line, sometimes oppositely; this follows in accordance with the various assumptions.

(24) And the additional ones. First, three by Charmandrus that are harmonious:

(2) If one end of a straight line given in magnitude be given, the other will touch a concave (circular) arc given in position.

(3) If straight lines from two given points should inflect and contain a given angle, their common point will touch a concave (circular) arc given in position.

(4) If the base of a triangular area given in magnitude should be given in position and magnitude, its vertex will touch a straight line given in position.

(25) Others are like this:

(5) If one end of a straight line given in magnitude and drawn parallel to some straight line given in position, should touch a straight line given in position, the other (end) too will touch a straight line given in position.

(6) If from a point to two straight lines given in position, whether parallel or intersecting, (straight lines) are drawn at given angles, either having a given ratio to one another, or with one of them plus that to which the other has a given ratio being given, the point will touch a straight line given in position.

(7) And if there be any number whatever of straight lines given in position, and straight lines be drawn to them from some point at given angles, and the (rectangle contained) by a given and a (line) drawn upon (one of them) plus the (rectangle contained) by a given and another (line) drawn upon (one of them) equals the (rectangle contained) by a given and another (line) drawn upon (one of them), and the rest similarly, the point will touch a straight line given in position.

(8) If from some point straight lines be drawn onto parallels given in position at given angles, and either cutting off straight lines as far as points given on them that have a (given) ratio (to each other) or containing a given area, or so that the given shapes (constructed) upon the (lines) drawn upon (them) or the excess of the shapes equals a given area, the point will touch a straight line given in position.

Prop. XXXV teor. XXV

Parallelogrammi posti sulla stessa base e tra le stesse parallele sono eguali tra loro[48]

Come abbiamo detto dei teoremi, che sono alcuni universali, altri particolari; e a quel modo che, distinguendoli, abbiamo aggiunto che sono alcuni semplici, altri composti, e abbiamo dimostrato che forma hanno gli uni e gli altri; così, secondo un'altra distinzione, diciamo che alcuni sono teoremi di luogo, altri no. E chiamo « di luogo » quelli ai quali capita di avere la stessa proprietà in un intero luogo; e chiamo « luogo » una posizione di una linea o di una superficie che effettua un'unica e medesima proprietà. Perché fra i teoremi di luogo alcuni sono costituiti in riferimento a linee, altri a superfici. E poiché fra le linee alcune sono piane, altre solide, — e sono piane quelle la cui generazione[49] nel piano è semplice come quella della retta, e solide sono quelle la cui generazione è resa manifesta da una sezione di una figura solida, come quelle dell'elica cilin-
395 drica e delle linee coniche — direi che fra i teoremi di luogo rife-
riti alle linee alcuni hanno un luogo piano, altri un luogo solido.

Ora il teorema qui proposto è di luogo, e di quelli di luogo riferiti alle linee, e piano; perché l'intero spazio tra le parallele è il luogo dei parallelogrammi costruiti sulla stessa base, che l'Autore degli *Elementi* dimostra essere eguali tra loro. Dei teoremi poi di luogo detti solidi valga come esempio questo: i parallelogrammi inscritti tra le asintoti e l'iperbole[50] sono eguali. Che l'iperbole sia una linea solida, è evidente: è infatti una sezione del cono.

Ora queste specie di teoremi Crisippo, come afferma Gemino, li assimilava alle idee. Come infatti queste racchiudono la ge-

48. 394,10 In questa proposizione e nella seguente XXXVI l'eguaglianza va intesa di superficie (equivalenza) non di forma (congruenza). V. in Enriques, I p. 115 sgg. il commento ai due teoremi.

49. 394,22 Accetto la correzione proposta da Ver Eecke, p. 337 n. 3 γένεσις in luogo di νόησις.

50. 395,11 Come spiega Ver Eecke, p. 338 n. 1, nello spazio compreso tra l'iperbole e le sue asintoti: prop. XII libro II delle *Coniche* d'Apollonio.

nerazione di un numero infinito di oggetti entro limiti determinati, così anche in questi teoremi la percezione di un numero infinito di figure avviene in luoghi determinati; e per opera di questa determinazione l'eguaglianza si fa manifesta, perché immaginando infiniti parallelogrammi costruiti sulla stessa base, se l'altezza delle parallele rimane la stessa, li mostra tutti eguali fra loro.

Pertanto il primo teorema di luogo che l'Autore degli *Elementi* ha registrato è quello qui esposto; e si direbbe che egli, col variare i teoremi secondo tutte le loro suddivisioni, non abbia voluto omettere, giustamente, nel suo trattato una tale specie di essi. Ma qui, poiché l'argomento riguarda le linee rette, presenta
luoghi piani riferiti a linee rette; mentre nel terzo libro, trat- 396
tando dei circoli e delle loro proprietà, c'insegnerà le circonferenze presenti nei teoremi di luogo piano.

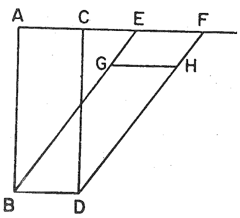
Tale è fra essi il teorema che dice: « Gli angoli inscritti nello stesso segmento di cerchio sono eguali fra loro » e « Gli angoli inscritti nel semicerchio sono retti ». Infatti angoli infiniti costruiti in una circonferenza sulla stessa base, si dimostrano tutti eguali; <e angoli compresi tra la base e il semicerchio sono angoli retti>. E quelle figure sono analoghe ai triangoli e ai parallelogrammi costruiti sulla stessa base <e tra le stesse parallele>[51].

Tale è dunque la specie dei teoremi che vanno attentamente studiati e che gli antichi matematici chiamavano di luogo. Ma potrà sembrare del tutto strano, a chi è inesperto in questa scienza, che parallelogrammi posti sulla stessa base <e fra le stesse parallele> siano fra loro eguali. Come infatti può essere, se la lunghezza[52] delle aree poste sulla stessa base cresce all'infinito — perché di quanto prolunghiamo le parallele, di altret-

51. 395,27 - 396,9 I due teoremi richiamati sono Euclide *Elem.* III 21 e 31; le integrazioni sono desunte da Barocius, non indispensabili, ma comunque aggiungono chiarezza. Cfr. Morrow.

52. 396,15 Coi termini μῆκος, e più sotto πλάτος, sono indicate le due dimensioni dei parallelogrammi compresi tra le parallele: πλάτος è la base e il lato opposto, che poggiano sulle parallele e restano sempre le stesse, μῆκος sono gli altri due lati, che, inizialmente perpendicolari alle parallele, si vanno via via spostando e allungando, dando l'impressione di ingrandire l'area; v. sotto 397, 10-12.

tanto possiamo aumentare le lunghezze dei parallelogrammi — come dunque, se avviene questo, possa mantenersi l'equivalenza delle aree, si potrà richiedere con ragione. Che se la larghezza è la stessa, — poiché la base è una sola — e la lunghezza è maggiore, come non è maggiore anche l'area? È dunque questo teorema e quello seguente sui triangoli, di quei teoremi che i
397 matematici chiamano paradossali; perché anche gli studiosi di matematica trattarono del luogo comune chiamato « paradosso », così come gli Stoici nei loro esempi[53], e pongono anche questo teorema fra quelli di tal genere. Colpisce dunque subito la maggior parte delle persone il fatto che la lunghezza accresciuta più volte non distrugge l'eguaglianza delle aree, quando la base resti la medesima. Ma parimente si deve dire che l'eguaglianza e la diseguaglianza degli angoli influisce al massimo sull'accrescimento o la diminuzione delle aree; perché se lunghezza e larghezza restano le stesse, di tanto diminuiamo l'area, di quanto rendiamo disuguali gli angoli. Bisogna dunque che aumenti la lunghezza, per conservare l'eguaglianza. Sia preso un parallelogrammo qualunque, ABCD, e si prolunghi il lato AC all'infinito. Il parallelogrammo sia eventualmente rettangolare, e sulla base BD se ne costruisca un altro, BEFD. Che dunque la lunghezza è aumentata, appare evidente; perché il lato BE è maggiore del lato AB, essendo l'angolo in A un angolo retto. Ma questo accrescimento avviene necessariamente; perché gli angoli del parallelogrammo BEFD sono diventati diseguali, gli uni acuti, gli altri ottusi. E questo è accaduto per il fatto che il lato BE si è come inclinato sul lato BD e ha ridotto l'area. Si



53. 397,2 δειγμάτων: forse da leggere δογμάτων? V. Morrow, p. 312 n. 78.

prenda infatti la retta BG eguale alla AB, e per G si conduca 398
la GH parallela alla BD. Allora la lunghezza del parallelogrammo BDGH è eguale alla lunghezza del ABCD, e la larghezza è la stessa, ma l'area è minore dell'area, perché l'area del parallelogrammo BDGH è minore di quella del BEFD. È appunto la diseguaglianza degli angoli che fa diminuire la superficie, mentre l'aumento della lunghezza, aggiungendo tanto quanto quella ha perduto, mantiene l'eguaglianza delle aree. Il limite poi dell'aumento della lunghezza è il luogo delle parallele. Infatti, se due parallelogrammi sono ambedue rettangoli, si dimostra che il quadrato è più grande dell'oblungo[54], e se ambedue sono equilateri, si dimostra che quello che è rettangolo è più grande del non rettangolo; perché la rettitudine degli angoli e l'eguaglianza dei lati hanno il massimo potere per l'accrescimento delle aree. Donde segue che il quadrato si dimostra il più grande, e il rombo il più piccolo di tutti gli isoperimetri.

Ma queste cose le dimostreremo in altri luoghi; perché sono più pertinenti alle proposizioni del secondo libro. Quanto al presente teorema, si deve sapere che Euclide, dicendo parallelogrammi eguali, intende dire le superfici, non i loro lati, — perché di queste si parla, delle superfici — e che ora per la prima volta, nella dimostrazione di questo teorema, Euclide fa menzione dei trapezi; per questo è chiaro che giustamente nelle definizioni[55] ci ha insegnato che cosa è il trapezio, cioè è qua- 399
drilatero come specie, ma non parallelogrammo; perché la figura che non ha lati ed angoli opposti eguali, esce dall'ordine dei parallelogrammi.

L'Autore degli *Elementi* dunque ha dimostrato la proposizione

54. 398,10 Che abbiano s'intende lo stesso perimetro, come aggiunge Barocius; così anche nell'esempio seguente. Su quel che segue cfr. T. C., *Two Questions*, p. 186 sg.

55. 398,25 Def. XXXIV 169,8 sgg.

FRAMMENTI GIOVANILI

REGOLE PER LA GUIDA DELL'INTELLIGENZA (1619-1630)

LA RICERCA DELLA VERITÀ MEDIANTE IL LUME NATURALE